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$\mathcal{H}_\infty$ Control of Nonlinear Systems: A Convex Characterization

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Abstract

The so-called nonlinear $\mathcal{H}_\infty$-control problem in state space is considered with an emphasis on developing machinery with promising computational properties. Both state feedback and output feedback $\mathcal{H}_\infty$-control problems for a class of nonlinear systems are characterized in terms of continuous positive definite solutions of algebraic nonlinear matrix inequalities which are convex feasibility problems. The issue of existence of solutions to these nonlinear matrix inequalities (NLMIs) is justified.

1 Introduction

Linear $\mathcal{H}_\infty$ control theory has been a very popular research area since it was originally formulated by Zames (cf. [8, 5, 7]) . The simplicity of the characterization of state space $\mathcal{H}_\infty$-control theory together with its clear connections with traditional methods in optimal control [7] have stimulated several attempts to generalize the linear $\mathcal{H}_\infty$ results in state space to nonlinear systems [26, 14, 2, 19]. We will use the accepted but unfortunate misnomer "nonlinear $\mathcal{H}_\infty$" to describe this research direction, which will be pursued further in this paper, with an eye toward computational issues.

Basically, in those generalizations, the necessary or sufficient conditions for the the $\mathcal{H}_\infty$-control problem to be solvable are characterized in terms of some Hamilton-Jacobi equations or inequalities [26, 14, 2, 19, 13, 27, 17]. Specially, an $\mathcal{H}_\infty$ output controller, which has separation structure, and a class of parametrized $\mathcal{H}_\infty$ controllers are designed based on the required solutions of Hamilton-Jacobi equations or inequalities [14, 19]. Whence, one of the major concerns in the state-space nonlinear $\mathcal{H}_\infty$-control theory is how to solve these Hamilton-Jacobi partial differential equations or inequalities, and progress along this line would be beneficial to applications of nonlinear $\mathcal{H}_\infty$-control theory. For example, van der Schaft [26] proposed an approach to approximate the solutions of Hamilton-Jacobi equations using Lukes' iteration method [21].

In this paper, we propose an alternative approach to the state-space nonlinear $\mathcal{H}_\infty$-control problem, and characterize the solutions in terms of convex conditions instead of the Hamilton-Jacobi equations or inequalities. This is motivated by the fact that, essentially, the linear $\mathcal{H}_\infty$-control problem can be characterized as a convex problem [23, 3]. We examine the convexity of the nonlinear $\mathcal{H}_\infty$-control problem, and deal with a class of nonlinear $\mathcal{H}_\infty$-control problem whose solvability

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conditions are convexly characterized as some algebraic **nonlinear matrix inequalities** (NLMIs). This makes the computation very appealing, since the solutions to this class of nonlinear $\mathcal{H}_\infty$ problems are based on the required solutions to the corresponding algebraic NLMIs, which can be solved via the convex optimization methods[3]. The analogical treatments in linear case, which are characterized in terms of **linear matrix inequalities** (LMIs), can be found in [20, 22, 18, 24, 9, 15].

The other features of the suggested approach are that the nonlinear system considered has few structural constraints; the system coefficient functions are just required to be continuous, no other smoothness is needed; our attention is not just paid to local solutions, the system is considered to evolve in some prescribed open convex set; in this framework, it is proved by using set-valued map machinery that if a concerned NLMI has solutions, then it has a continuous solution which is required for the $\mathcal{H}_\infty$-control problem to be solvable; the algebraic NLMIs are in fact the state-dependent LMIs, therefore, the existing convex optimization methods for solving LMIs can be used in the practical computation for solving NLMIs. Unfortunately, unlike the linear case, the solution of the NLMIs by themselves are not sufficient to guarantee the existence of the required controller, and the computational implications of the required additional constraints on the NLMI solutions are not clear. This issue is discussed more in the body of the paper.

This paper is organized as follows: In section 2, some background material related to the $\mathcal{L}_2$-gains analysis is provided; the NLMI characterization of $\mathcal{L}_2$-gains is given. In section 3, the $\mathcal{H}_\infty$-control problem is stated. In section 4, the $\mathcal{H}_\infty$-control problem for state-feedback is considered; both static feedback and dynamic feedback are examined. In section 5, the main results of this paper, i.e., solutions to the output feedback $\mathcal{H}_\infty$-control problem, are given; the solvability of this problem is convexly characterized by two NLMIs and a *coupling condition*. In section 6, the existence of the solutions to these NLMIs is confirmed under some mild conditions. Some required technical material is reviewed in the appendix.

The following conventions are made in this paper. $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+ := [0, \infty) \subset \mathbb{R}$. $\mathbb{R}^n$ is n-dimensional real Euclidean space; $\|\cdot\|$ stands for the **Euclidean norm**. For $\mathcal{B}_r$, it is understood to be the open ball in some Euclidean space with some radius $r > 0$ which is measured by Euclidean norm. $\mathbf{X}$ (or $\mathbf{X}_o$) is the state set which is a convex open subset of some Euclidean space and contains the origin. $\mathbb{R}^{n\times m}$ ($\mathbb{C}^{n\times m}$) is the set of all $n \times m$ real (complex) matrices. The transpose of some matrix $M \in \mathbb{R}^{n\times n}$ is denoted by M^T. By $P > 0$ ($P \geq 0$) for some Hermitian matrix $P \in \mathbb{R}^{n\times n}$ or ($\mathbb{C}^{n\times m}$) we mean that the matrix is (semi-)positive definite. A function is said to be of class $\mathbf{C}^k$ if it is continuously differentiable k times; so $\mathbf{C}^0$ stands for the class of continuous functions.

2 Preliminaries: $\mathcal{L}_2$-Gains and Strong $\mathcal{H}_\infty$-Performances

In this section, some background material about $\mathcal{L}_2$-gain analysis of nonlinear systems is provided. The reader is referred to Willems [28] and van der Schaft [26] for more details.

2.1 $\mathcal{L}_2$-Gain Analysis

Consider the following affine nonlinear time-invariant (NLTI) system:

$$G : \begin{cases} \dot{x} = A(x)x + B(x)w \\ z = C(x)x + D(x)w \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is state vector, $w \in \mathbb{R}^p$ and $z \in \mathbb{R}^q$ are input and output vectors, respectively. We will assume A, B, C, D are $\mathbf{C}^0$ matrix-valued functions of suitable dimensions. From now on we will assume the system evolves on a convex open subset $\mathbf{X} \subset \mathbb{R}^n$ containing the origin. Thus, $0 \in \mathbb{R}^n$ is the equilibrium of the system with $w = 0$. The state transition function $\phi : \mathbb{R}^+ \times \mathbf{X} \times \mathbb{R}^p \to \mathbf{X}$ is so defined that $x = \phi(T, x_0, w^*)$ means that system G evolves from initial state x_0 to state x in time T under the control action w^*.

Definition 2.1 *(i) The system G (or $[A(x), B(x)]$) is* **reachable** *from* 0 *if for all $x \in \mathbf{X}$, there exist $T \in R^+$ and $w^*(t) \in \mathcal{L}_2[0,T]$ such that $x = \phi(T, 0, w^*)$;*

(ii) The system G (or $[C(x), A(x)]$) is (zero-state) **detectable** *if for all $x \in \mathbf{X}$, $h(\phi(t,x,0)) = 0 \Rightarrow \phi(t,x,0) \to 0$ as $t \to \infty$; it is (zero-state)* **observable** *if for all $x \in \mathbf{X}$, $h(\phi(t,x,0)) = 0 \Rightarrow \phi(t,x,0) = 0$ for all $t \in \mathbb{R}^+$.*

Definition 2.2 *The system G with initial state $x(0) = 0$ is said to have $\mathcal{L}_2$-***gain less than or equal to** *γ for some $\gamma > 0$ if*

$$\int_0^T \|z(t)\|^2 \, dt \le \gamma^2 \int_0^T \|w(t)\|^2 \, dt \tag{2}$$

for all $T \ge 0$ and $w(t) \in \mathcal{L}_2[0,T]$, and $z(t) = h(\phi(t,0,w(t)) + k(\phi(t,0,w(t))w(t)$.

In the following discussion, we only consider the case $\gamma = 1$ without loss of generality. Define

$$V_a(x) := \sup_{w \in \mathcal{L}_2[0,\infty), x(0)=x} - \int_0^\infty (\|w(t)\|^2 - \|z(t)\|^2) dt. \tag{3}$$

Note that $V_a(x) \ge 0$ for all $x \in \mathbf{X}$, and if the system has $\mathcal{L}_2$-gain ≤ 1, then $V_a(0) = 0$. We will assume $V_a(0) = 0$ from now on.

As pointed out by Willems [28], $V_a(x) < \infty$ if and only if there exists a function $V : \mathbf{X} \to \mathbb{R}^+$ with $V(0) = 0$, which is called **storage function** in [28], such that:

$$\mathcal{D}_I(V, x, w) := V(x) - V(x_0) - \int_0^T (\|w(t)\|^2 - \|z(t)\|^2) dt \le 0 \tag{4}$$

where $x = \phi(T, x_0, w(t))$ and $w(t) \in \mathcal{L}_2[0,T]$, and $V_a(\cdot)$ is also a solution. Moreover, the solutions to (4) form a convex set, and any solution $V(x) \ge 0$ for $x \in \mathbf{X}$ with $V(0) = 0$ satisfies $V(x) \ge V_a(x)$.

Lemma 2.1 (Willems [28])

Suppose system G is reachable from 0. It has $\mathcal{L}_2$-gain ≤ 1 if and only if $V_a(x) < \infty$, for all $x \in \mathbf{X}$.

Thus, if the system is reachable from 0, then $\mathcal{L}_2$-gain ≤ 1 if and only if $V_a(x) \geq 0$ is well-defined for all $x \in \mathbf{X}$, and there exists a solution $V(x) \geq 0$ to the above (4). Now assume $V_a(x)$ and $V(x)$ are of class $\mathbf{C}^1$ and can be writen as

$$V_a(x) = x^T Q_a x + r_a(x), \qquad V(x) = x^T Q x + r(x)$$

for some $Q_a, Q \geq 0$ and $\mathbf{C}^1$ functions $r_a, r : \mathbf{X} \rightarrow \mathbb{R}$ satisfying

$$\lim_{x \to 0} \frac{|r_a(x)|}{\|x\|^2} = 0, \qquad \lim_{x \to 0} \frac{|r(x)|}{\|x\|^2} = 0.$$

Since $V_a(0) = 0, V(0) = 0$, there are $\mathbf{C}^0$ matrix-valued functions $P_a, P : \mathbf{X} \rightarrow \mathbb{R}^{n \times n}$ such that

$$\frac{\partial V_a}{\partial x}(x) = 2x^T P_a^T(x), \qquad \frac{\partial V}{\partial x}(x) = 2x^T P^T(x).$$

Now for matrix-valued function $P : \mathbf{X} \rightarrow \mathbb{R}^{n \times n}$, let $R(x) := I - D^T(x)D(x) > 0$, we define

$$\begin{aligned}\mathcal{H}(P, x) := P^T(x)(A(x) - B(x)R^{-1}(x)D^T(x)C(x)) + (A^T(x) - C^T(x)D(x)R^{-T}(x)B^T(x))P(x) + \\ + P^T(x)B(x)R^{-1}(x)B^T(x)P(x) + C^T(x)(I - D(x)D^T(x))^{-1}C(x). \qquad (5)\end{aligned}$$

The following standard result characterizes a class of nonlinear systems having $\mathcal{L}_2$-gain ≤ 1.

Lemma 2.2 *Consider a system G with $R(x) := I - D^T(x)D(x) > 0$ for all $x \in \mathbf{X}$; suppose G has $\mathcal{L}_2$-gain ≤ 1, and $V_a, V : \mathbf{X} \rightarrow \mathbb{R}^+$ are defined as above.*
i) If $P_a : \mathbf{X} \rightarrow \mathbb{R}^{n \times n}$ is such that $\frac{\partial V_a}{\partial x}(x) = 2x^T P_a^T(x)$, then $x^T \mathcal{H}(P_a, x)x = 0$;
ii) If $P : \mathbf{X} \rightarrow \mathbb{R}^{n \times n}$ is such that $\frac{\partial V}{\partial x}(x) = 2x^T P^T(x)$, then $x^T \mathcal{H}(P, x)x \leq 0$.

Proof See [26, 19]. □

Recall that $V : \mathbf{X} \rightarrow \mathbb{R}^+$ is locally positive-definite if there exists $r > 0$ such that for $x \in \mathcal{B}_r$, $V(x) = 0 \Rightarrow x = 0$; it is globally positive-definite if $V(x) = 0 \Rightarrow x = 0$, and $\lim_{x \to \infty} V(x) = \infty$. It is easy to see that the converse results in the above lemma are also true.

Proposition 2.3 *Suppose there is a $\mathbf{C}^0$ matrix-valued function $P : \mathbf{X} \rightarrow \mathbb{R}^{n \times n}$, such that $\mathcal{H}(P, x) \leq 0$ for all $x \in \mathbf{X}$, and there is a non-negative function $V(x) \geq 0$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P^T(x)$, then the concerned system has L_2-gain ≤ 1.*

Proof Note that $\mathcal{H}(P, x) \leq 0$ implies

$$\begin{aligned}A^T(x)P(x) + P^T(x)A(x) \leq -C^T(x)C(x) + \\ -(P^T(x)B(x) + C^T(x)D(x))(I - D^T(x)D(x))^{-1}(B^T(x)P(x) + D^T(x)C(x)). \qquad (6)\end{aligned}$$

Take V as defined in the statement, then

$$\dot{V}(x) = \frac{\partial V}{\partial x}(x)(A(x)x + B(x)w) = 2x^T P^T(x)(A(x)x + B(x)w)$$

$$= x^T(P^T(x)A(x) + A^T(x)P(x))x + 2x^T P^T(x)B(x)w \tag{7}$$

$$\leq \|w(t)\|^2 - \|z(t)\|^2 - \left\|w(t) + R^{-1}(x)D^T(x)C(x)x + R^{-1}(x)B^T(x)P(x)x\right\|^2$$

The latter inequality follows by replacing $P^T(x)A(x) + A^T(x)P(x)$ in (6) into (7) and reorganizing it. Therefore,

$$\dot{V}(x) - (\|w(t)\|^2 - \|z(t)\|^2) \leq 0.$$

Take the integral from $t = 0$ to $t = T$, the above inequality implies inequality (4), which in turn implies the system has $\mathcal{L}_2$-gain ≤ 1 since $V(x) \geq 0$. □

Note that in the above characterization the solution P to $\mathcal{H}(P,x) \leq 0$ is not required to be (semi-)positive definite, even symmetric. However, the solutions can be chosen as positive definite in the cases of interest in the present paper; this is justified in the next subsection. If $P(x) = P^T(x) > 0$ for $x \in \mathbf{X}$, then $V(x)$ with $V(0) = 0$ satisfying $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ is (locally) positive definite (see lemma 8.3).

The following statement, which is due to Hill and Moylan (see [26]), establishes the relationship between finite gain (stability) and asymptotic stability.

Lemma 2.4 *(i) Suppose the system G with $u = 0$ is asymptotically stable at 0, then any $V(x)$ with $V(0) = 0$ satisfying $\frac{\partial V}{\partial x}(x) = 2x^T P^T(x)$ such that $\mathcal{H}(P,x) \leq 0$ is such that $V(x) \geq 0$ for all $x \in \mathbf{X}$.*

(ii) Assume system G is zero-state detectable. If there is a positive definite function $V(x)$ with $V(0) = 0$, $\frac{\partial V}{\partial x}(x) = 2x^T P^T(x)$ such that $\mathcal{H}(P,x) \leq 0$, then the system G with $u = 0$ is asymptotically stable at 0.

Now there comes up the main result of this section, which characterizes the $\mathcal{L}_2$-gain of the system.

Theorem 2.5 *Consider the system G given by (1), suppose $I - D^T(x)D(x) > 0$. Given any $\mathbf{C}^0$ matrix-valued function $P : \mathbf{X} \rightarrow \mathbb{R}^{n \times n}$, the following inequalities are equivalent.*

(i) P satisfies $\mathcal{H}(P,x) \leq 0$;

(ii) P satisfies

$$\mathcal{M}(P,x) := \begin{bmatrix} A^T(x)P(x) + P^T(x)A(x) + C^T(x)C(x) & P^T(x)B(x) + C^T(x)D(x) \\ B^T(x)P(x) + D^T(x)C(x) & D^T(x)D(x) - I \end{bmatrix} \leq 0; \tag{8}$$

(iii) P satisfies

$$\hat{\mathcal{M}}(P,x) := \begin{bmatrix} A^T(x)P(x) + P^T(x)A(x) & P^T(x)B(x) & C^T(x) \\ B^T(x)P(x) & -I & D^T(x) \\ C(x) & D(x) & -I \end{bmatrix} \leq 0. \tag{9}$$

In addition, suppose the considered system is asymptotically stable with $w = 0$. If there are a $\mathbf{C}^0$ matrix-valued function $P : \mathbf{X} \rightarrow \mathbb{R}^{n \times n}$ satisfying any of the above inequalities and a function $V : \mathbf{X} \rightarrow \mathbb{R}$ such that $\frac{\partial V}{\partial x}(x) = 2xP^T(x)$, then the system has $\mathcal{L}_2$-gain ≤ 1.

Inequalities (8) and (9) are actually a state-dependent linear matrix inequalities. We call them **nonlinear matrix inequalities** (NLMIs) here to emphasize that they are used to deal with nonlinear systems. In section 6, we shall consider the computational issue in solving NLMIs. It should be emphasized that the existence of a $\mathbf{C}^0$ matrix-valued function $P : \mathbf{X} \rightarrow \mathbb{R}^{n\times n}$ which satisfies any of the above inequalities is not enough to guarantee the system to have $\mathcal{L}_2$-gain ≤ 1; in this theorem, it is additionally required that there exists a function $V : \mathbf{X} \rightarrow \mathbb{R}$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P^T(x)$. (See lemma 8.2 for a characterization of a class of matrix-valued function $P : \mathbf{X} \rightarrow \mathbb{R}^{n\times n}$ which satisfies this additional requirement.)

Proof Reorganize the left hand side of the inequality $\mathcal{H}(P,x) \leq 0$,

$$\begin{aligned}\mathcal{H}(P,x) &= A^T(x)P(x) + P^T(x)A(x) + C^T(x)C(x) + \\ &+ (P^T(x)B(x) + C^T(x)D(x))(I - D^T(x)D(x))^{-1}(B^T(x)P(x) + D^T(x)C(x)) \leq 0. \quad (10)\end{aligned}$$

So by using the standard result about Schur complements, we have $\mathcal{H}(P,x) \leq 0$ if and only if $\mathcal{M}(P,x) \leq 0$ or $\hat{\mathcal{M}}(P,x) \leq 0$, since it is assumed $I - D^T(x)D(x) > 0$.

If the $\mathbf{C}^0$ matrix-valued function P satisfies any of the above inequalities and there is a function $V : \mathbf{X} \rightarrow \mathbb{R}$ with $V(0) = 0$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P^T(x)$. By preceding lemma, $V(x) \geq 0$. Therefore, by proposition 2.3, the system has $\mathcal{L}_2$-gain ≤ 1. □

Remark 2.1 *If there is a $\mathbf{C}^0$ matrix-valued matrix P_0 such that $\mathcal{M}(P_0, x) < 0$ for $x \in \mathbf{X}$, then by continuity of $\mathcal{M}$ with respect to x, there is another $\mathbf{C}^0$ matrix-valued matrix P such that $\mathcal{M}(P,x) < 0$ and $\frac{\partial V}{\partial x}(x) = 2x^T P^T(x)$ for some $\mathbf{C}^1$ function $V : \mathcal{B}_d \rightarrow \mathbb{R}^+$ for some $d > 0$. In fact, a natural choice is $P = P_0(0)$, and $V(x) = x^T P x$. Therefore, in this paper, the requirement that $\frac{\partial V}{\partial x}(x) = 2x^T P^T(x)$ for a $\mathbf{C}^0$ solution $P(x)$ to any NLMI and a $\mathbf{C}^1$ function $V(x)$ is not very strong in a local sense.*

2.2 Solutions to NLMIs and Strong $\mathcal{H}_\infty$-Performances

Consider the NLMI (8) or (9). By theorem 2.5, there exists a matrix valued function $P : \mathbf{X} \rightarrow \mathbb{R}^{n\times n}$ such that $\mathcal{M}(P,x) \leq 0$ or $\hat{\mathcal{M}}(P,x) \leq 0$ if and only if there exists a matrix valued function $Q : \mathbf{X} \rightarrow \mathbb{R}^{n\times n}$ with $Q(x) \geq 0$ for all $x \in \mathbf{X}$ such that

$$\mathcal{H}(P,x) + Q(x) = 0.$$

Let $R(x) := I - D^T(x)D(x) > 0$, define a state-dependent Hamiltonian $H : \mathbf{X} \rightarrow \mathbb{R}^{2n\times 2n}$ as

$$H(x) := \begin{bmatrix} A(x) & 0 \\ -C^T(x)C(x) - Q(x) & -A^T(x) \end{bmatrix} + \begin{bmatrix} B(x) \\ -C^T(x)D(x) \end{bmatrix} R^{-1}(x) \begin{bmatrix} D^T(x)C(x) & B^T(x) \end{bmatrix} \quad (11)$$

Now for fixed $x \in \mathbf{X}$, λ is an eigenvalue of $H(x)$ if and only if $-\bar{\lambda}$ is. Hence, there must be at least n eigenvalues for $H(x)$ in half plane $\mathrm{Re}(s) \leq 0$. Suppose that we choose an n-dimensional invariant subspace, denoted by $\mathcal{X}_-(H(x))$, corresponding to the n eigenvalues in $\mathrm{Re}(s) \leq 0$ and

$$\mathcal{X}_-(H(x)) = Span \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad (12)$$

where $X_1, X_2 \in \mathbb{C}^{n\times n}$, and

$$H(x)\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} T_X . Re(\lambda_i(T_X)) \le 0 \forall i \tag{13}$$

If X_1 is invertible, set $X := X_2 X_1^{-1}$, suppose in addition X is Hermitian, then the map

$$\bar{Ric} : H(x) \mapsto X \tag{14}$$

is well defined [10], and its domain $dom(\bar{Ric})$ will be taken to consist of Hamiltonian matrices with the above properties. So we have the following result which is essentially from [10, lemma 2.4] and theorem 2.5.

Theorem 2.6 *$\mathcal{M}(P, x) \le 0$ has non-negative definite solutions $P(x) \ge 0$ if and only if the state-dependent Hamiltonian $H : \mathbf{X} \to \mathbb{R}^{2n\times 2n}$ defined in (11) for some matrix-valued function $Q : \mathbf{X} \to \mathbb{R}^{n\times n}$ with $Q(x) \ge 0$ for all $x \in \mathbf{X}$ is in $dom(\bar{Ric})$, i.e. $H(x) \in dom(\bar{Ric})$ for each $x \in \mathbf{X}$. Moreover, $\bar{Ric}(H(x)) \ge 0$ is such a solution. In addition, if for each $x \in \mathbf{X}$,*

$$\bigcap_{i=0}^{n-1} \ker(C(x)A^i(x)) = \emptyset,$$

this solution is positive definite, i.e., $\bar{Ric}(H(x)) > 0$.

The above theorem implies that under the condition $H(x) \in dom(\bar{Ric})$ for each $x \in \mathbf{X}$, which is not restrictive at all, the NLMI $\mathcal{M}(P, x) \le 0$ has non-negative definite solutions. In section 6, we will further show that such solutions can be chosen to be continuous in the cases of interest in this paper. A nice convex property for NLMIs is stated by the following proposition whose proof is easy and omitted here.

Proposition 2.7 *The $\mathbf{C}^0$ solutions $P : \mathbf{X} \to \mathbb{R}^{n\times n}$ to NLMI $\mathcal{M}(P, x) \le 0$ form a convex set; the subset of all $\mathbf{C}^0$ non-negative definite solution $P = P^T : \mathbf{X} \to \mathbb{R}^{n\times n}$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ for some function $V : \mathbf{X} \to \mathbb{R}$ is convex; the subset of all $\mathbf{C}^1$ positive definite solutions $P^T(x) = P(x) > 0$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ for some function $V : \mathbf{X} \to \mathbb{R}$ is also convex.*

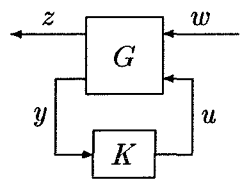
It is noted that by lemma 8.3, the $\mathbf{C}^1$ function $V : \mathbf{X} \to \mathbb{R}$ which satisfies $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ for some positive definite solutions $P^T(x) = P(x) > 0$ and $V(0) = 0$ is positive definite on $\mathbf{X}$. Now we close this section by defining a stronger $\mathcal{H}_\infty$-performance.

Definition 2.3 *The concerned system is said to have* **strong $\mathcal{H}_\infty$-performance** *if there is a $\mathbf{C}^0$ positive definite function $P(x) = P^T(x) > 0$ which satisfies any of inequalities (8) and (9) for all $x \in \mathbf{X}$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ for some $\mathbf{C}^1$ function $V : \mathbf{X} \to \mathbb{R}$.*

So if system has a strong $\mathcal{H}_\infty$-performance, by lemma 8.3, proposition 2.3 and theorem 2.5, it has $\mathcal{L}_2$-gain ≤ 1.

3 $\mathcal{H}_\infty$-Control Problem

The feedback configuration for the $\mathcal{H}_\infty$-control synthesis problem is depicted as follows,



where G is the nonlinear plant with two sets of inputs: the exogenous disturbance inputs w and the control inputs u, and two sets of outputs: the measured outputs y and the regulated outputs z. And K is the controller to be designed. It is assumed the feedback configuration is **well-posed**. Both G and K are nonlinear time-invariant and can be realized as control-affine state-space equations:

$$G : \begin{cases} \dot{x} &= A(x)x + B_1(x)w + B_2(x)u \\ z &= C_1(x)x + D_{11}(x)w + D_{12}(x)u \\ y &= C_2(x)x + D_{21}(x)w + D_{22}(x)u \end{cases}$$

where $A, B_i, C_i, D_{ij} \in \mathbf{C}^0$; x, w, u, z, and y are assumed to have dimensions n, p_1, p_2, q_1, and q_2, respectively.

$$K : \begin{cases} \dot{\xi} &= \hat{A}(\xi)\xi + \hat{B}(\xi)y \\ u &= \hat{C}(\xi)\xi + \hat{D}(\xi)y \end{cases}$$

with $\hat{A}, \hat{B}, \hat{C}, \hat{D} \in \mathbf{C}^0$. It is assumed that the feedback system evolves in $(x, \xi) \in \mathbf{X} \times \mathbf{X}_o$, where $\mathbf{X}$ and $\mathbf{X}_o$ are open convex sets and contain the origins. The initial states for both plant and controller are $x(0) = 0$ and $\xi(0) = 0$.

In this paper, we shall consider the following version of $\mathcal{H}_\infty$-control problem.

(Strong) $\mathcal{H}_\infty$-Control Problem: Find a feedback controller K (or a class controllers) if any, such that the closed-loop system has **strong $\mathcal{H}_\infty$-performance**. In this case, the feedback system has $\mathcal{L}_2$-gain ≤ 1, i.e.

$$\int_0^T (\|w(t)\|^2 - \|z(t)\|^2)dt \geq 0;$$

for all $T \in \mathbb{R}^+$.

The controllers to be sought in solving the above $\mathcal{H}_\infty$-Control Problem is called **strong $\mathcal{H}_\infty$-controllers**. Note that the stability issue is not explicitly touched here, as it is guaranteed by the observability assumption (see [14, 19]). In the following, we will mainly consider two cases:

- The state x is directly available to the control action u, so the problem is called state feedback $\mathcal{H}_\infty$-control problem. We will examine both static and dynamic state feedback.
- Only the output y can be directly measured, so in this case the problem is called (output feedback) $\mathcal{H}_\infty$-control problem. We will mainly consider dynamic feedback.

4 State Feedback $\mathcal{H}_\infty$-Control Problem

In this section, we consider the (strong) $\mathcal{H}_\infty$-control problem that the state x is directly measured. We will first consider the static feedback case and show that the state feedback $\mathcal{H}_\infty$-control problem is solvable by static feedbacks if and only if some NLMI has required solutions. Next, we will consider the dynamic state feedback $\mathcal{H}_\infty$-control problem, and show that the dynamic feedback can not do better than static feedback can when the strong $\mathcal{H}_\infty$ control problem is considered.

In this section, we consider the following system,

$$G_{SF}: \left\{ \begin{array}{rcllll} \dot{x} &=& A(x)x &+& B_1(x)w &+& B_2(x)u \\ z &=& C_1(x)x &+& D_{11}(x)w &+& D_{12}(x)u \\ y &=& x \end{array} \right. \tag{15}$$

with $A, B_i, C_1, D_{ij} \in \mathbf{C}^0$. The state x, disturbance w, control input u, and regulated out put z have dimensions of n, p_1, p_2, and p_1, respectively; and $n + q_1 - p_2 \geq 0$. We assume that system evolves in $\mathbf{X}$, $rank \begin{bmatrix} B_2(x) \\ D_{12}(x) \end{bmatrix} = p_2$ and $D_{11}(x)D_{11}(x) < I$ for $x \in \mathbf{X}$.

4.1 Static State Feedback

Consider the system G_{SF}. Suppose the controller $u = F(x)x$ is such that the closed loop system

$$\left\{ \begin{array}{rcll} \dot{x} &=& (A(x) + B_2(x)F(x))x &+& B_1(x)w \\ z &=& (C_1(x) + D_{12}(x)F(x))x &+& D_{11}(x)w \end{array} \right.$$

has strong $\mathcal{H}_\infty$-performance. By the definition 2.3, there is a $\mathbf{C}^0$ positive definite matrix-valued function $X = X^T : \mathbf{X} \to \mathbb{R}^{n \times n}$ such that $\frac{\partial V}{\partial x}(x) = 2x^T X(x)$ for some $\mathbf{C}^1$ function $V : \mathbf{X} \to \mathbb{R}^+$ and the following NLMI holds.

$$\begin{bmatrix} (A^T(x) + F^T(x)B_2^T(x))X(x) + X(x)(A(x) + B_2(x)F(x)) & X(x)B_1(x) & C_1^T(x) + F^T(x)D_{12}^T \\ B_1^T(x)X(x) & -I & D_{11}^T(x) \\ C_1(x) + D_{12}(x)F(x) & D_{11}(x) & -I \end{bmatrix} \leq$$

We use the notation $\mathcal{M}_{SF}(X, F, x)$ to represent the left hand side of the above inequality. Define

$$T(x) = \begin{bmatrix} X^{-1}(x) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

which is well-defined since $X(x) > 0$. Note that $\mathcal{M}_{SF}(X, F, x) \leq 0$ if and only if

$$T^T(x)\mathcal{M}_{SF}(X, F, x)T(x) \leq 0.$$

Let $X(x) = P^{-1}(x)$, which is of class $\mathbf{C}^0$, then

$$T^T(x)\mathcal{M}_{SF}(P, F, x)T(x) = \tilde{\mathcal{M}}_{SF}(X, x) + \tilde{X}^T(x)F^T(x)\tilde{B}(x) + \tilde{B}^T(x)F(x)\tilde{X}(x) \leq 0 \tag{16}$$

where

$$\tilde{\mathcal{M}}_{SF}(X,x) := \begin{bmatrix} X^T(x)A^T(x) + A(x)X(x) & B_1(x) & X^T(x)C_1^T(x) \\ B_1^T(x) & -I & D_{11}^T(x) \\ C_1(x)X(x) & D_{11}(x) & -I \end{bmatrix},$$

$$\tilde{X}(x) = \begin{bmatrix} X(x) & 0 & 0 \end{bmatrix}, \qquad \tilde{B}(x) = \begin{bmatrix} B_2^T(x) & 0 & D_{12}^T(x) \end{bmatrix}$$

Now by lemma 8.6, it follows that there is a solution $F(x)$ for (16) if and only if

$$X_\perp^T(x)\tilde{\mathcal{M}}_{SF}(X,x)X_\perp(x) \leq 0 \tag{17}$$

$$\tilde{B}_\perp^T(x)\tilde{\mathcal{M}}_{SF}(X,x)\tilde{B}_\perp(x) \leq 0 \tag{18}$$

for some $X_\perp(x)$ such that $span(X_\perp(x)) = \mathcal{N}(\tilde{X}(x))$ and $\tilde{B}_\perp(x)$ with $span(\tilde{B}_\perp(x)) = \mathcal{N}(\tilde{B}(x))$. Here $\mathcal{N}(B(x))$ for some matrix-valued function $B(x)$ stands for the distribution which annihilates all of the row vectors of $B(x)$ (see (67)).

Notice that (17) is guaranteed by the assumption $I - D_{11}^T(x)D_{11}(x) > 0$; (18) is actually written as

$$\tilde{B}_\perp^T(x) \begin{bmatrix} X^T(x)A^T(x) + A(x)X(x) & B_1(x) & X^T(x)C_1^T(x) \\ B_1^T(x) & -I & D_{11}^T(x) \\ C_1(x)X(x) & D_{11}(x) & -I \end{bmatrix} \tilde{B}_\perp(x) \leq 0 \tag{19}$$

Whence, using Schur complement arguments, we can conclude the following theorem.

Theorem 4.1 *The strong static state feedback $\mathcal{H}_\infty$-control problem is solvable if and only if there is a $\mathbf{C}^0$ matrix-valued function $X(x) = X^T(x) > 0$ with $\frac{\partial V}{\partial x}(x) = 2x^T X^{-1}(x)$ for some $\mathbf{C}^1$ function $V : \mathbf{X} \to \mathbb{R}^+$ such that for all $x \in \mathbf{X}$, the following NLMI holds:*

$$B_\perp^T(x) \begin{bmatrix} X(x)A^T(x) + A(x)X(x) + B_1(x)B_1^T(x) & X(x)C_1^T(x) + B_1(x)D_{11}^T(x) \\ C_1(x)X(x) + D_{11}(x)B_1^T(x) & D_{11}(x)D_{11}^T(x) - I \end{bmatrix} B_\perp(x) \leq 0 \tag{20}$$

with $B_\perp : \mathbf{X} \to \mathbb{R}^{(n+q_1)\times(n+q_1-p_2)}$ such that $span(B_\perp(x)) = \mathcal{N}(B(x))$, where

$$B(x) := \begin{bmatrix} B_2^T(x) & D_{12}^T(x) \end{bmatrix},$$

for all $x \in \mathbf{X}$.

4.2 Dynamic State-Feedback vs. Static State-feedback

Consider system G_{SF}. Suppose the strong $\mathcal{H}_\infty$-control problem is solved by the following dynamic feedback

$$K_d : \left\{ \begin{array}{lllll} \dot{\xi} & = & A_d(\xi)\xi & + & B_d(\xi)x \\ u & = & C_d(\xi)\xi & + & D_d(\xi)x \end{array} \right.$$

with $A_d, B_d, C_d, D_d \in \mathbf{C}^0$. Assume $\xi \in \mathbf{X}_o$. The closed loop system with state $x_c = \begin{bmatrix} x \\ \xi \end{bmatrix}$ is as follows

$$\left\{ \begin{array}{lllll} \dot{x}_c & = & A_c(x_c)x_c & + & B_c(x_c)w \\ z & = & C_c(x_c)x_c & + & D_c(x_c)w \end{array} \right.$$

where $A_c(x_c) = A^a(x) + B_2^a(x)F_c(\xi)$, $B_c(x_c) = B_1^a(x)$, $C_c(x_c) = C_1^a(x) + D_{12}^a(x)F_c(\xi)$, and $D_c(x_c) = D_{11}^a(x)$ with

$$A^a(x) := \begin{bmatrix} A(x) & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_1^a(x) := \begin{bmatrix} B_1(x) \\ 0 \end{bmatrix}, \qquad B_2^a(x) := \begin{bmatrix} B_2(x) & 0 \\ 0 & I \end{bmatrix},$$

$$C_1^a(x) := \begin{bmatrix} C_1(x) & 0 \end{bmatrix}, \qquad D_{11}^a(x) := D_{11}(x), \qquad D_{12}^a(x) := \begin{bmatrix} D_{12}(x) & 0 \end{bmatrix},$$

and

$$F_c(\xi) := \begin{bmatrix} D_d(\xi) & C_d(\xi) \\ B_d(\xi) & A_d(\xi) \end{bmatrix}.$$

So the original problem is transformed into a static feedback $\mathcal{H}_\infty$-control problem, whence the previous results apply. It follows that there is a $\mathbf{C}^0$ positive definite matrix-valued function $X_c(x_c) = X_c^T(x_c) > 0$ holds for all $x_c \in \mathbf{X}{\times}\mathbf{X}$ such that the following NLMI holds for all $x_c \in \mathbf{X}{\times}\mathbf{X}_o$

$$\breve{B}_\perp^T(x) \begin{bmatrix} X_c(x_c)(A^a(x))^T + A^a(x)X_c(x_c) & B_1^a(x) & X_c(x)(C_{11}^a(x))^T \\ (B_1^a(x))^T & -I & D_{11}^T(x) \\ C_1^a(x)X_c(x_c) & D_{11}(x) & -I \end{bmatrix} \breve{B}_\perp(x) \leq 0 \tag{21}$$

for all matrix-valued function $\breve{B}_\perp(x)$ such that $span(\breve{B}_\perp(x)) \in \mathcal{N}(\breve{B}(x))$ with

$$\breve{B}(x) = \begin{bmatrix} B_2(x) & 0 & D_{12}(x) \\ 0 & I & 0 \end{bmatrix}.$$

Moreover, there is a positive definite function $V_c : \mathbf{X} \times \mathbf{X}_o \to \mathbb{R}^+$ such that $\frac{\partial V_c}{\partial x_c}(x_c) = 2x_c^T X_c^{-1}(x_c)$ for all $(x, \xi) \in \mathbf{X} \times \mathbf{X}_o$. Now the following assumption is made.

Assumption 4.2 *There is a* $\mathbf{C}^1$ *function* $\phi : x \mapsto \xi$ *with* $\phi(0) = 0$ *such that* $\frac{\partial V_c}{\partial \xi}(x, \xi)|_{\xi=\phi(x)} = 0$ *with* $(x, \xi) \in \mathbf{X} \times \mathbf{X}_o$.

Note that the above assumption is not very restrictive (see remark 5.2).

Now the NLMI (21) holds for all $(x, \xi) \in \mathbf{X} \times \mathbf{X}_o$, so it holds for $(x, \phi(x)) \in \mathbf{X} \times \mathbf{X}_o$. Define $X : \mathbf{X} \to \mathbb{R}^{n \times n}$ is such that

$$X_c(x, \phi(x)) = \begin{bmatrix} X(x) & X_1^T(x) \\ X_1(x) & X_0(x) \end{bmatrix}$$

Thus, $X(x)$ is also positive definite and of class $\mathbf{C}^0$. In this case, (21) becomes

$$\tilde{B}_\perp^T(x) \begin{bmatrix} X(x)A^T(x) + A(x)X(x) & B_1(x) & X(x)C_1^T(x) \\ B_1^T(x) & -I & D_{11}^T(x) \\ C_1(x)X(x) & D_{11}(x) & -I \end{bmatrix} \tilde{B}_\perp(x) \leq 0 \tag{22}$$

for all $\tilde{B}_\perp(x)$ with $span(\tilde{B}_\perp(x)) \in \mathcal{N}(\tilde{B}(x))$ where $\tilde{B}(x) := \begin{bmatrix} B_2^T(x) & 0 & D_{12}^T(x) \end{bmatrix}$.

Next, let's further consider the possibility for solving the $\mathcal{H}_\infty$-control by static state feedback. Now $\frac{\partial V_c}{\partial x_c}(x_c) = 2x_c^T X_c^{-1}(x_c)$ implies $\frac{\partial V_c}{\partial x_c}(x_c) X_c(x_c) = 2x_c^T$, or

$$\begin{bmatrix} \frac{\partial V_c}{\partial x}(x, \xi) & \frac{\partial V_c}{\partial \xi}(x, \xi) \end{bmatrix} X_c(x, \xi) = 2 \begin{bmatrix} x^T & \xi^T \end{bmatrix} \tag{23}$$

Now take $\xi = \phi(x)$, under assumption 4.2, (23) implies $\frac{\partial V_c}{\partial x}(x, \phi(x)) X(x) = 2x^T$. Define $V(x) := V_c(x, \phi(x))$, then $V(x)$ is positive definite such that

$$\frac{\partial V}{\partial x}(x) = 2x^T X^{-1}(x). \tag{24}$$

Combine (22) with (24), we can conclude that the $\mathcal{H}_\infty$-control problem is indeed solvable in terms of static feedback. So we have the following theorem.

Theorem 4.3 *Consider the state feedback $\mathcal{H}_\infty$-control problem. If it is solvable in terms of the dynamic feedback, then there is a $\mathbf{C}^0$ positive definite matrix-valued function $X : \mathbf{X} \to \mathbb{R}^{n \times n}$ such that the following NLMI holds*

$$B_\perp^T(x) \begin{bmatrix} X(x)A^T(x) + A(x)X(x) + B_1(x)B_1^T(x) & X(x)C_1^T(x) + B_1(x)D_{11}^T(x) \\ C_1(x)X(x) + D_{11}(x)B_1^T(x) & D_{11}^T(x)D_{11}(x) - I \end{bmatrix} B_\perp(x) \leq 0$$

with $B_\perp^T : \mathbf{X} \to \mathbb{R}^{(n+q_1)\times(n+q_1-p_2)}$ such that $\mathcal{N}(B(x)) = span(B_\perp(x))$, where

$$B(x) := \begin{bmatrix} B_2^T(x) & D_{12}^T(x) \end{bmatrix},$$

for $x \in \mathbf{X}$. In addition, under assumption 4.2, the $\mathcal{H}_\infty$ control problem can also be solved in terms of static state feedback.

4.3 Output Injection

Analogically, the strong $\mathcal{H}_\infty$-control problem can also be solved in terms of output injections. The output-injection structure is

$$G_{OI}:\left\{\begin{array}{ccccccc} \dot{x} & = & A(x)x & + & B_1(x)w & + & u \\ z & = & C_1(x)x & + & D_{11}(x)w & & \\ y & = & C_2(x)x & + & D_{21}(x)w & & \end{array}\right. \tag{25}$$

with $A, B_1, C_i, D_{ij} \in \mathbf{C}^0$. Suppose x, w, u, z and y have dimensions of n, p_1, n, q_1 and q_2, respectively; and $n + p_1 - q_2 \geq 0$. We assume that system evolves in $\mathbf{X}$; $rank \left[\begin{array}{cc} C_2(x) & D_{21}(x) \end{array}\right] = q_2$ and $D_{11}(x)D_{11}(x) < I$ for $x \in \mathbf{X}$. The solvability condition is also characterizes in terms of some NLMI; and the strong $\mathcal{H}_\infty$-control problem is solved by a static output if and only if it is solved by dynamic output injection. This fact is stated in the following theorem without proof.

Theorem 4.4 *The strong output injection $\mathcal{H}_\infty$-control problem is solvable if and only if there is a $\mathbf{C}^0$ matrix-valued function $Y(x) = Y^T(x) > 0$ with $\frac{\partial U}{\partial x}(x) = 2x^TY(x)$ for some $\mathbf{C}^1$ function $U : \mathbf{X}\rightarrow\mathbb{R}^+$ such that for all $x \in \mathbf{X}$, the following NLMI holds:*

$$C_\perp^T(x)\left[\begin{array}{cc} A^T(x)Y(x)+Y(x)A(x)+C_1^T(x)C_1(x) & Y(x)B_1(x)+C_1^T(x)D_{11}(x) \\ B_1^T(x)Y(x)+D_{11}^T(x)C_1(x) & D_{11}^T(x)D_{11}(x)-I \end{array}\right]C_\perp(x) \leq 0 \tag{26}$$

with $C_\perp : \mathbf{X}\rightarrow\mathbb{R}^{(n+p_1)\times(n+p_1-q_2)}$ such that $\mathcal{N}(C(x)) = span(C_\perp(x))$, where $C(x) := \left[\begin{array}{cc} C_2(x) & D_{21}(x) \end{array}\right]$, for all $x \in \mathbf{X}$.

5 Output Feedback $\mathcal{H}_\infty$-Control Problem

In this section, we will consider the general strong $\mathcal{H}_\infty$-control problem; the system to be considered is

$$G:\left\{\begin{array}{ccccccc} \dot{x} & = & A(x)x & + & B_1(x)w & + & B_2(x)u \\ z & = & C_1(x)x & + & D_{11}(x)w & + & D_{12}(x)u \\ y & = & C_2(x)x & + & D_{21}(x)w & + & D_{22}(x)u \end{array}\right. \tag{27}$$

where $A, B_i, C_i, D_{ij} \in \mathbf{C}^0$; x, w, u, z, and y are assumed to have dimensions n, p_1, p_2, q_1, and q_2, respectively; $n + p_1 \geq q_2$ and $n + q_1 \geq p_2$. Suppose the system (27) evolves in $\mathbf{X}$ which is a convex open subset of $\mathbb{R}^n$ and contains the origin; assume $rank \left[\begin{array}{c} B_2(x) \\ D_{12}(x) \end{array}\right] = p_2$ and $rank \left[\begin{array}{cc} C_1(x) & D_{21}(x) \end{array}\right] = q_2$, and $D_{11}(x)D_{11}^T(x) < I$ for all $x \in \mathbf{X}$.

5.1 Necessary Conditions

Suppose the strong $\mathcal{H}_\infty$-controller is also of control-affine form:

$$K : \left\{ \begin{array}{lllll} \dot{\xi} & = & \hat{A}(\xi)\xi & + & \hat{B}(\xi)y \\ u & = & \hat{C}(\xi)\xi & + & \hat{D}(\xi)y \end{array} \right.$$

with $\hat{A}, \hat{B}, \hat{C}, \hat{D} \in \mathbf{C}^0$. Suppose $\xi \in \mathbf{X}_o \subset \mathbb{R}^{n_d}$. The closed loop system evolves in $(x, \xi) \in \mathbf{X} \times \mathbf{X}_o$. We shall also assume that $I - \hat{D}(\xi)D_{22}(x)$ is invertible for all $(x, \xi) \in \mathbf{X} \times \mathbf{X}_o$ to assure the well-posedness of the feedback structure. Now take $x_c = \begin{bmatrix} x \\ \xi \end{bmatrix}$ to be the state of the closed loop system; define $R(x_c) := (I - \hat{D}(\xi)D_{22}(x))^{-1}$ for $(x, \xi) \in \mathbf{X} \times \mathbf{X}_o$. The feedback system has the following description:

$$\left\{ \begin{array}{lllll} \dot{x}_c & = & A_c(x_c)x_c & + & B_c(x_c)w \\ z & = & C_c(x_c)x_c & + & D_c(x_c)w \end{array} \right.$$

where

$$A_c(x_c) := \begin{bmatrix} A(x) + B_2(x)R(x_c)\hat{D}(\xi)C_2(x) & B_2(x)R(x_c)\hat{C}(\xi) \\ \hat{B}(\xi)(I + D_{22}(x)R(x_c)\hat{D}(\xi))C_2(x) & \hat{A}(\xi) + \hat{B}(\xi)D_{22}(x)R(x_c)\hat{C}(\xi) \end{bmatrix},$$

$$B_c(x_c) := \begin{bmatrix} B_1(x) + B_2(x)R(x_c)\hat{D}(\xi)D_{21}(x) \\ \hat{B}(\xi)(D_{21}(x) + D_{22}(x)R(x_c)\hat{D}(\xi))D_{21}(x) \end{bmatrix},$$

$$C_c(x_c) := \begin{bmatrix} C_1(x) + D_{12}(x)R(x_c)\hat{D}(\xi)C_2(x) & D_{12}(x)R(x_c)\hat{C}(\xi) \end{bmatrix},$$

$$D_c(x_c) := D_{11}(x) + D_{12}(x)R(x_c)\hat{D}(\xi)D_{21}(x).$$

The main theorem of this section is stated as follows.

Theorem 5.1 *Suppose there is a solution to the output feedback (strong) $\mathcal{H}_\infty$ control problem, then there are two $\mathbf{C}^0$ symmetrical matrix-valued functions $X, Y : \mathbf{X} \rightarrow \mathbb{R}^{n \times n}$, which are positive definite on $\mathbf{X}$, such that for all $x \in \mathbf{X} \subset \mathbb{R}^{n \times n}$:*

$$(i)\ B_\perp^T(x) \begin{bmatrix} X(x)A^T(x) + A(x)X(x) + B_1(x)B_1^T(x) & X(x)C_1^T(x) + B_1(x)D_{11}^T(x) \\ C_1(x)X(x) + D_{11}(x)B_1^T(x) & D_{11}(x)D_{11}^T(x) - I \end{bmatrix} B_\perp(x) \leq 0 \tag{28}$$

with $B_\perp : \mathbf{X} \rightarrow \mathbb{R}^{(n+q_1)\times(n+q_1-p_2)}$ such that $\mathcal{N}(B(x)) = span(B_\perp(x))$, where $B(x) := \begin{bmatrix} B_2^T(x) & D_{12}^T(x) \end{bmatrix}$,

$$(ii)\ C_\perp^T(x) \begin{bmatrix} A^T(x)Y(x) + Y(x)A(x) + C_1^T(x)C_1(x) & Y(x)B_1(x) + C_1^T(x)D_{11}(x) \\ B_1^T(x)Y(x) + D_{11}^T(x)C_1(x) & D_{11}^T(x)D_{11}(x) - I \end{bmatrix} C_\perp(x) \leq 0 \tag{29}$$

with $C_\perp : \mathbf{X} \rightarrow \mathbb{R}^{(n+p_1)\times(n+p_1-q_2)}$ such that $\mathcal{N}(C(x)) = span(C_\perp(x))$, where $C(x) := \begin{bmatrix} C_2(x) & D_{21}(x) \end{bmatrix}$,

$$(iii) \begin{bmatrix} X(x) & I \\ I & Y(x) \end{bmatrix} \geq 0.$$

Remark 5.1 *It is noted that all couples $(X(x), Y(x))$ satisfying the inequalities (i), (ii) and (iii) form a convex set. Therefore, theorem 5.1 provides a convex characterization to the necessary conditions for the strong output feedback $\mathcal{H}_\infty$-control problem to be solvable.*

Proof Define

$$A^a(x) := \begin{bmatrix} A(x) & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1^a(x) := \begin{bmatrix} B_1(x) \\ 0 \end{bmatrix}, \quad B_2^a(x_c) := \begin{bmatrix} B_2(x) & 0 \\ \hat{B}(\xi) D_{22}(x) & I \end{bmatrix},$$

$$C_1^a(x) := \begin{bmatrix} C_1(x) & 0 \end{bmatrix}, \quad D_{11}^a(x) := D_{11}(x), \quad D_{12}^a(x) := \begin{bmatrix} D_{12}(x) & 0 \end{bmatrix},$$

$$C_2^a(x) := \begin{bmatrix} C_2(x) & 0 \\ 0 & I \end{bmatrix}, \quad D_{21}^a(x) := \begin{bmatrix} D_{21}(x) \\ 0 \end{bmatrix}, \quad D_{22}^a(x) := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$F_c(x_c) := \begin{bmatrix} R(x_c)\hat{D}(\xi) & R(x_c)\hat{C}(\xi) \\ \hat{B}(\xi) & \hat{A}(\xi) \end{bmatrix}.$$

Thus

$$A_c = A^a(x) + B_2^a(x_c) F_c(x_c) C_2^a(x), \quad B_c(x_c) = B_1^a(x) + B_2^a(x_c) F_c(x_c) D_{21}^a(x),$$

$$C_c(x_c) = C_1^a + D_{12}^a F_c(x_c) C_2^a(x), \quad D_c(x_c) = D_{11}^a(x) + D_{12}^a(x) F_c(x_c) D_{21}^a(x).$$

Since the feedback system has strong $\mathcal{H}_\infty$-performance, by definition 2.3, there is a $\mathbf{C}^0$ positive definite matrix-valued function $P_c(x_c)$ on $\mathbf{X} \times \mathbf{X}_o$ such that

$$\mathcal{M}_c(P_c, x_c) := \begin{bmatrix} A_c^T(x_c)P_c(x_c) + P_c(x_c)A_c(x_c) & P_c(x_c)B_c(x_c) & C_c^T(x_c) \\ B_c^T(x_c)P_c(x_c) & -I & D_c^T(x_c) \\ C_c(x_c) & D_c(x_c) & -I \end{bmatrix} \le 0. \tag{30}$$

Re-organizing the left hand side of the above NLMI yields

$$\mathcal{M}_c(X_c, x_c) = \mathcal{M}_a(P_c, x_c) + \tilde{C}^T(x) F_c^T(x_c) \tilde{B}(x_c) T_c(x_c) + T_c^T(x_c) \tilde{B}^T(x_c) F_c(x_c) \tilde{C}(x) \le 0 \tag{31}$$

where

$$\mathcal{M}_a(P_c, x_c) := \begin{bmatrix} (A_a(x))^T P_c(x_c) + P_c(x_c)A^a(x) & P_c(x_c)B_1^a(x) & (C_1^a(x))^T \\ (B_1^a(x))^T P_c(x_c) & -I & (D_{11}^a(x))^T \\ C_1^a(x) & D_{11}^a(x) & -I \end{bmatrix} \le 0.$$

$$\tilde{B}(x_c) := \begin{bmatrix} (B_2^a(x_c))^T & 0 & (D_{12}^a(x))^T \end{bmatrix}, \quad \tilde{C}(x) := \begin{bmatrix} C_2^a(x) & D_{21}^a(x) & 0 \end{bmatrix}$$

and

$$T_c(x_c) = \begin{bmatrix} P_c(x_c) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

So it follows from lemma 8.6 that (31) holds if and only if the following two inequalities hold (see lemma 8.6):

$$\tilde{B}_\perp^T(x_c) T_c^{-T}(x_c) \mathcal{M}_a(P_c, x_c) T_c^{-1}(x_c) \tilde{B}_\perp(x_c) \le 0, \tag{32}$$

$$\tilde{C}_\perp^T(x)\mathcal{M}_a(P_c, x_c)\tilde{C}_\perp(x) \leq 0 \tag{33}$$

for all $\tilde{B}_\perp(x_c)$ with $span(\tilde{B}_\perp(x_c)) \in \mathcal{N}(\tilde{B}(x_c))$ and $\tilde{C}_\perp(x)$ with $span(\tilde{C}_\perp(x)) \in \mathcal{N}(\tilde{C}(x))$.

Now we consider (32), notice that $\mathcal{N}(\tilde{B}(x_c)) = \mathcal{N}(\check{B}(x))$ for

$$\check{B}(x) := \begin{bmatrix} B_2^T(x) & 0 & 0 & D_{12}^T(x) \\ 0 & I & 0 & 0 \end{bmatrix}.$$

Thence, (32) holds if and only if

$$\check{B}_\perp^T(x)T_c^{-T}(x_c)\mathcal{M}_a(P_c, x_c)T_c^{-1}(x_c)\check{B}_\perp(x) \leq 0, \tag{34}$$

for all $\check{B}_\perp(x)$ with $span(\check{B}_\perp(x)) \in \mathcal{N}(\check{B}(x))$. On the other hand, notice that

$$T_c^{-T}(x_c)\mathcal{M}_a(P_c, x_c)T_c^{-1}(x_c) = \begin{bmatrix} P_c^{-1}(x_c)(A^a(x))^T + A^a(x)P_c^{-1}(x_c) & B_1^a(x) & P_c^{-1}(x)(C_{11}^a(x))^T \\ (B_1^a(x))^T & -I & D_{11}^T(x) \\ C_1^a(x)P_c^{-1}(x_c) & D_{11}(x) & -I \end{bmatrix}$$

Since $P_c(x_c) = P_c(x, \xi)$ is invertible on $\mathbf{X} \times \mathbf{X}_o$, assume $X(x) = X^T(x) \in \mathbb{R}^{n \times n}$, which is positive definite and of class $\mathbf{C}^0$ on $\mathbf{X}$, is such that

$$P_c^{-1}(x, \phi(x)) = \begin{bmatrix} X(x) & X_1^T(x) \\ X_1(x) & X_0(x) \end{bmatrix}, \tag{35}$$

for some continuously differentiable function $\phi : x \mapsto \xi$ in $\mathbf{X}$ such that $\phi(\mathbf{X}) \subset \mathbf{X}_o$ (for example ϕ can be chosen as $\phi(x) = 0$). Therefore, by the arguments of Schur complements, (34), i.e. (32) implies

$$B_\perp^T(x)\begin{bmatrix} X(x)A^T(x) + A(x)X(x) + B_1(x)B_1^T(x) & X(x)C_1^T(x) + B_1(x)D_{11}^T(x) \\ C_1(x)X(x) + D_{11}(x)B_1^T(x) & D_{11}(x)D_{11}^T(x) - I \end{bmatrix} B_\perp(x) \leq 0$$

with $B_\perp : \mathbf{X} \to \mathbb{R}^{(n+q_1)\times(n+q_1-p_2)}$ such that $\mathcal{N}(B(x)) = span(B_\perp(x))$, where $B(x) := \begin{bmatrix} B_2^T(x) & D_{12}^T(x) \end{bmatrix}$.

Thus, the first part is proved. Now if we take $Y(x) \in \mathbb{R}^{n\times n}$, which is of class $\mathbf{C}^0$, such that

$$P_c(x, \phi(x)) = \begin{bmatrix} Y(x) & Y_1^T(x) \\ Y_1(x) & Y_0(x) \end{bmatrix}, \tag{36}$$

then (33) implies

$$C_\perp^T(x)\begin{bmatrix} A^T(x)Y(x) + Y(x)A(x) + C_1^T(x)C_1(x) & Y(x)B_1(x) + C_1(x)D_{11}(x) \\ B_1^T(x)Y(x) + D_{11}^T(x)C_1(x) & D_{11}^T(x)D_{11}(x) - I \end{bmatrix} C_\perp(x) \leq 0$$

with $C_\perp : \mathbf{X} \to \mathbb{R}^{(n+p_1)\times(n+p_1-q_2)}$ such that $\mathcal{N}(C(x)) = span(C_\perp(x))$, where $C(x) := \begin{bmatrix} C_2(x) & D_{21}(x) \end{bmatrix}$.

Now what is left un-proved is the last part. But by lemma 8.5, (35) and (36) hold if and only if

$$\begin{bmatrix} X(x) & I \\ I & Y(x) \end{bmatrix} \geq 0.$$

This concludes the proof. $\square$

5.2 Output Feedback and State Feedback

In this section, we further show that if the $\mathcal{H}_\infty$-control problem is solvable by output feedback, then it is also solvable by static state feedback and static output injection. This statement is not trivial, and is not the conclusion of the above theorem. This point will be clear from the following discussion.

Suppose the output feedback strong $\mathcal{H}_\infty$-control problem for the given system (27) is solvable, then there is a $\mathbf{C}^0$ positive definite matrix-valued function $P_c(x_c)$ such that (30) holds. Moreover, there is a positive definite function $V_c(x_c)$ such that

$$\frac{\partial V_c}{\partial x_c}(x_c) = 2x_c^T P_c(x_c)$$

We make the following assumption.

Assumption 5.2 *There is a $\mathbf{C}^1$ function $\phi : x \mapsto \xi$ with $\phi(0) = 0$ such that $\frac{\partial V_c}{\partial \xi}(x,\xi)|_{\xi=\phi(x)} = 0$ with $(x,\xi) \in \mathbf{X} \times \mathbf{X}_o$.*

Remark 5.2 *This assumption as well as assumption 4.2 is not surprising. In fact, many dynamical controllers are observer-like-based [2, 14, 19, 16]. So the states x, ξ of a plant and its controller have a relation $\xi = \phi(x)$ for some $\mathbf{C}^1$ function $\phi : x \mapsto \xi$ with $\phi(0) = 0$ if the initial states satisfy $\xi(0) = \phi(x(0))$ and the disturbance is not imposed. The Lyapunov function for the closed loop system can be taken as $V_c(x,\xi) = V(x) + U(\xi - \phi(x))$ where V and U are Lyapunov functions of the state-feedback system and the observer. Thence, $\frac{\partial V_c}{\partial \xi}(x,\xi) = \frac{\partial U}{\partial e}(e)|_{e=\xi-\phi(x)}$. Now if $e = 0$, i.e. $\xi = \phi(x)$, then $\frac{\partial V_c}{\partial \xi}(x,\xi)|_{\xi=\phi(x)} = \frac{\partial U}{\partial e}(e)|_{e=0} = 0$. Therefore, V_c satisfies the assumption.*

Now from the proof of the last theorem, it follows that (30) implies that there is $X(x) = X^T(x) \in \mathbb{R}^{n\times n}$, which is positive definite and of class $\mathbf{C}^0$ on $\mathbf{X}$, such that

$$P_c^{-1}(x,\phi(x)) = \begin{bmatrix} X(x) & X_1^T(x) \\ X_1(x) & X_0(x) \end{bmatrix},$$

for some continuously differentiable function $\phi : x \mapsto \xi$ on $\mathbf{X}$, and the following NLMI holds,

$$B_\perp^T(x) \begin{bmatrix} X(x)A^T(x) + A(x)X(x) + B_1(x)B_1^T(x) & X(x)C_1^T(x) + B_1(x)D_{11}^T(x) \\ C_1(x)X(x) + D_{11}(x)B_1^T(x) & D_{11}(x)D_{11}^T(x) - I \end{bmatrix} B_\perp(x) \leq 0$$

with $B_\perp : \mathbf{X} \to \mathbb{R}^{(n+q_1)\times(n+q_1-p_2)}$ such that $\mathcal{N}(B(x)) = span(B_\perp(x))$, where $B(x) := \begin{bmatrix} B_2^T(x) & D_{12}^T(x) \end{bmatrix}$.

Now we take the function ϕ as in assumption 5.2, define $V(x) := V_c(x,\phi(x))$. By the similar argument in subsection 4.2, it follows that $V(x)$ is positive definite and

$$\frac{\partial V}{\partial x}(x) = 2x^T X^{-1}(x). \tag{37}$$

So it can be concluded that the $\mathcal{H}_\infty$-control problem is indeed solvable in terms of static feedback. Thence, we have the following proposition.

Proposition 5.3 *If the strong $\mathcal{H}_\infty$-control problem is solvable in terms of the output feedback, then under assumption 5.2, it can also be solved in terms of static state feedback.*

Similar argument applies to output injection problem. Taking $\phi : x \mapsto \xi$ as $\phi(x) = 0$. Define a positive definite matrix-valued function $Y(x) = Y^T(x) \in \mathbb{R}^{n \times n}$ such that

$$P_c(x, 0) = \begin{bmatrix} Y(x) & Y_1^T(x) \\ Y_1(x) & Y_0(x) \end{bmatrix},$$

Define $U(x) = V_c(x, 0)$, which is positive definite, then $\frac{\partial V_c}{\partial x_c}(x_c) = 2x_c^T P_c(x_c)$ implies

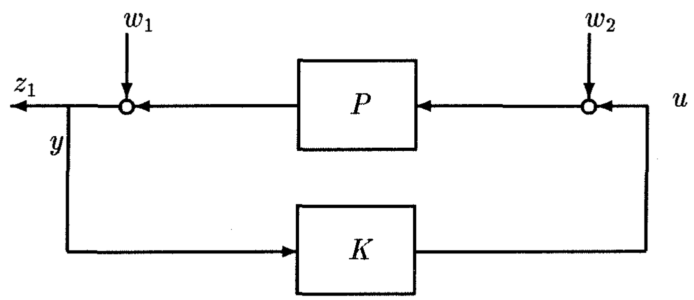
$$\frac{\partial U}{\partial x}(x) = 2x^T Y(x). \tag{38}$$

Thus, by combining (29) and (38), we also have the following result.

Proposition 5.4 *If the strong $\mathcal{H}_\infty$-control problem is solvable in terms of the output feedback, then it can also be solved in terms of static output injection.*

5.3 An Example

In this subsection, we will examine an example which is from [6, 19]. The basic diagram is as follows.



Where P is the nonlinear plant; K is the controller to be designed such that the output z_1 is regulated; y is the measured output, based on which the control action u is produced; w_2 is the disturbance from the actuator; and w_1 is the noise from the sensor. The **control problem** is to design the controller K such that the influence of the noises w_1 and w_2 on the regulated output z_1 can be reduced to the minimal with the reasonable effort (control action should not be too large).

To formulate this problem, all the signals are considered in space $\mathcal{L}_2[0, \infty)$; define the optimal achievable $\mathcal{L}_2$-gain for this feedback system to be

$$\gamma^* := \inf_K\{\gamma : \int_0^T (\|z_1\|^2 + \|u\|^2)dt \le \gamma^2 \int_0^T (\|w_1\|^2 + \|w_2\|^2)dt, \forall T \in \mathbb{R}^+; w_1, w_2 \in \mathcal{L}_2[0, \infty)\} \tag{39}$$

The $\mathcal{H}_\infty$-control problem in this setting is formulated as: Give $\gamma \ge \gamma^*$, find K such that

$$\int_0^T (\|z_1\|^2 + \|u\|^2)dt \le \gamma^2 \int_0^T (\|w_1\|^2 + \|w_2\|^2)dt, \forall T \in \mathbb{R}^+$$

In this example , the plant has the following realization:

$$\begin{cases} \dot{x} = e^x(w_2 + u) \\ z_1 = x + w_1 \\ y = x + w_1 \end{cases}$$

It is known from [6] that the optimal achievable $\mathcal{L}_2$-gain for this feedback system is $\gamma^* = \sqrt{2}$. Take

$$w := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad z := \frac{1}{\sqrt{2}} \begin{bmatrix} z_1 \\ u \end{bmatrix}$$

as input and output vectors, then the state-space realization is

$$\begin{cases} \dot{x} &= & & \begin{bmatrix} 0 & e^x \end{bmatrix} w &+ & e^x u \\ z &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} x &+ & \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} w &+ & \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} u \\ y &= x &+ & \begin{bmatrix} 1 & 0 \end{bmatrix} w & & \end{cases}$$

Therefore, there should be a controller K such that the closed loop system satisfies

$$\int_0^T \|z\|^2 \, dt \le \int_0^T \|w\|^2 \, dt, \forall T \in \mathbb{R}^+$$

Whence, the three conditions (i), (ii) and (iii) in theorem 5.1 should be satisfied. We now verify this.

We first consider NLMI in condition (i), which is as follows

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \sqrt{2}e^x \end{bmatrix} \begin{bmatrix} e^{2x} & \frac{1}{\sqrt{2}}X(x) & 0 \\ \frac{1}{\sqrt{2}}X(x) & -1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & \sqrt{2}e^x \end{bmatrix} \le 0,$$

it is equivalent to

$$\begin{bmatrix} -1/2 & -\frac{1}{\sqrt{2}}X(x) \\ \frac{1}{\sqrt{2}}X(x) & -e^{2x} \end{bmatrix} \le 0.$$

Thus, all positive definite solutions satisfy

$$X(x) \le e^x. \tag{40}$$

The NLMI in condition (ii) is as follows

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & e^x Y(x) \\ 1/2 & -1/2 & 0 \\ e^x Y(x) & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \le 0,$$

which is equivalent to

$$\begin{bmatrix} -1 & e^x Y(x) \\ e^x Y(x) & -1 \end{bmatrix} \le 0.$$

Therefore, all positive definite solutions satisfy

$$Y(x) \le e^{-x} \tag{41}$$

We then take two special solutions in (40) and (41) as

$$X(x) = e^x, \qquad Y(x) = e^{-x}.$$

Then $X(x)Y(x) = 1$, which implies condition (iii), i.e.,

$$\begin{bmatrix} e^x & 1 \\ 1 & e^{-x} \end{bmatrix} \geq 0.$$

Actually, the optimal $\mathcal{L}_2$-gain $\sqrt{2}$ is achieved by constant feedback $K = -1$. This can be checked using theorem 2.5.

6 Solutions to Nonlinear Matrix Inequalities

In this section, we'd like to consider the computational issue in solving the NLMIs that appear in this paper, i.e., (8), (9), (19), (20), (28), and (29). Technically, the material in this section is also of independent interest.

6.1 Existence of Continuous Solutions

In section 2.2, we justified the existence of positive definite solutions to NLMIs. Since the solvability for each strong $\mathcal{H}_\infty$-control problem requires that the positive definite solutions to the corresponding NLMIs be **continuous**, a natural question is that, is there such a solution to a NLMI if it has positive definite solutions? In this section, we will justify this, i.e., we will consider **the existence of continuous solutions** to the given NLMIs provided that there are some solutions but we don't know if they are continuous. We tackle this problem using the set-valued map machinery [1]. For a brief review about set-valued maps, consult Appendix 8.3.

Let $\mathbf{S}(\mathbb{R}^{n\times n})$ be the set of all $(n\times n)$-dimensional symmetric matrices, and $\mathbf{P}(\mathbb{R}^{n\times n})$ be its subset of semi-positive definite matrices. Both are Banach spaces with the Euclidean norm. Let $\mathbf{X}$ be an open subset $\mathbb{R}^n$ with $0 \in \mathbf{X}$, consider a general matrix-valued map $\mathcal{M} : \mathbf{P}(\mathbb{R}^{n\times n}) \times \mathbf{X} \rightarrow \mathbf{S}(\mathbb{R}^{m\times m})$, which is continuous and satisfies

$$\mathcal{M}(\alpha P_1 + (1-\alpha)P_2, x) = \alpha\mathcal{M}(P_1, x) + (1-\alpha)\mathcal{M}(P_2, x). \tag{42}$$

for some $\alpha \in \mathbb{R}$. Consider the following matrix inequality.

$$\mathcal{M}(P, x) \leq 0. \tag{43}$$

Note that **all of the NLMIs discussed in this paper are in this matrix inequality class.**

Next, define two set-valued functions $\mathcal{F}, \check{\mathcal{F}} : \mathbf{X} \rightsquigarrow \mathbf{P}(\mathbb{R}^{n\times n})$ as

$$\mathcal{F}(x) := \{P : \mathcal{M}(P, x) \leq 0\} \tag{44}$$

$$\check{\mathcal{F}}(x) := \{P : \mathcal{M}(P, x) < 0\} \tag{45}$$

It is assumed that both set-valued functions $\mathcal{F}$ and $\check{\mathcal{F}}$ are strict, i.e., $\mathrm{Dom}(\mathcal{F}) = \mathbf{X}$ and $\mathrm{Dom}(\check{\mathcal{F}}) = \mathbf{X}$.

It is observed that the set-valued maps $\mathcal{F}$ and $\check{\mathcal{F}}$ have following properties.

Lemma 6.1 *(i) The set-valued function* $\mathcal{F} : \mathbf{X} \rightsquigarrow \mathbf{P}(\mathbb{R}^{n\times n})$ *takes values as closed convex subset of* $\mathbf{P}(\mathbb{R}^{n\times n})$.

(ii) $\check{\mathcal{F}} : \mathbf{X} \rightsquigarrow \mathbf{P}(\mathbb{R}^{n\times n})$ *takes values as convex subset of* $\mathbf{P}(\mathbb{R}^{n\times n})$.

Proof We only prove part (i), since (ii) can be proved similarly.

Take $x \in \mathbf{X}$, consider the value $\mathcal{F}(x)$. We first examine the **convexity of** $\mathcal{F}(x)$. Let $P_1, P_2 \in \mathcal{F}(x)$, then by (44),

$$\mathcal{M}(P_1, x) \leq 0, \qquad \mathcal{M}(P_2, x) \leq 0.$$

For any $\alpha \in [0,1]$, by (42)

$$\mathcal{M}(\alpha P_1 + (1-\alpha)P_2, x) = \alpha\mathcal{M}(P_1, x) + (1-\alpha)\mathcal{M}(P_2, x) \leq 0.$$

so $\alpha P_1 + (1-\alpha)P_2 \in \mathcal{F}(x)$.

Next, consider the **closedness of** $\mathcal{F}(x)$. Let $\{P_n\} \subset \mathcal{F}(x)$ be a sequence which converges to $P_0 \in \mathbf{P}(\mathbb{R}^{n\times n})$. We need to prove $P_0 \in \mathcal{F}(x)$. In fact, $P_n \in \mathcal{F}(x)$ implies

$$0 \geq \mathcal{M}(P_n, x), \forall n \in \{1, 2, 3, \cdots\}$$

Since $\mathcal{M}$ is continuous, it follows by taking limits on both sides that

$$0 \geq \lim_{n\to\infty} \mathcal{M}(P_n, x) = \mathcal{M}(\lim_{n\to\infty} P_n, x) = \mathcal{M}(P_0, x)$$

which implies $P_0 \in \mathcal{F}(x)$. □

Lemma 6.2 *(i) The set-valued function $\mathcal{F} : \mathbf{X} \rightsquigarrow \mathbf{P}(\mathbb{R}^{n\times n})$ is lower semi-continuous.*

(ii) $\check{\mathcal{F}} : \mathbf{X} \rightsquigarrow \mathbf{P}(\mathbb{R}^{n\times n})$ is lower semi-continuous.

Proof The part (i) is proved here, part (ii) follows similarly.

Take $x_0 \in \mathbf{X}$, let $P_0 \in \mathcal{F}(x_0)$, and $\epsilon > 0$. Now it is sufficient to show that *there is a neighborhood* $\mathbf{N}(x_0)$ *of* $x_0 \in \mathbf{X}$ *such that for all* $x \in \mathbf{N}(x_0)$, *there exists a* $P_x \in \mathcal{F}(x) \cap \{P : \|P - P_0\| < \epsilon\}$.

By the choices of x_0 and P_0, it follows that

$$\mathcal{M}(P_0, x_0) \leq 0.$$

Now we **claim** that there is a $P_x \in \mathbf{P}(\mathbb{R}^{n\times n})$ which satisfies $\|P_x - P_0\| < \epsilon$ such that

$$\mathcal{M}(P_x, x_0) < 0. \tag{46}$$

In fact, since $\check{\mathcal{F}}$ is strict, there is $P_1 \in \check{\mathcal{F}}(x_0)$ such that

$$\mathcal{M}(P_1, x_0) < 0,$$

moreover, $P_1 \in \mathcal{F}(x_0)$; on the other hand, the convexity of $\mathcal{F}(x_0)$ implies that there exists $P_x = \alpha P_0 + (1-\alpha)P_1$ satisfying $\|P_x - P_0\| < \epsilon$ with some $\alpha \in [0,1)$ such that $P_x \in \mathcal{F}(x_0)$. Moreover,

$$\mathcal{M}(P_x, x_0) = \mathcal{M}(\alpha P_0 + (1-\alpha)P_1, x_0) = \alpha\mathcal{M}(P_0, x_0) + (1-\alpha)\mathcal{M}(P_1, x_0) < 0,$$

which confirms the claim. On the other hand, since $\mathcal{M}(P, x)$ is continuous, for fixed P_x which satisfies (46), there is some $\delta > 0$, such that if $x \in \mathbf{N}(x_0) := \{x : \|x - x_0\| < \delta\} \subset \mathbf{X}$, then

$$\mathcal{M}(P_x, x) \leq 0.$$

so $P_x \in \mathcal{F}(x)$. So there indeed exists a $P_x \in \mathcal{F}(x) \cap \{P : \|P - P_0\| < \epsilon\}$. □

Now the main result of this section is stated as follows:

Theorem 6.3 *Suppose the matrix inequality $\mathcal{M}(P,x) < 0$ has a positive semi-definite solution $P = P^T \geq 0$ for each $x \in \mathbf{X}$, then there exists a $\mathbf{C}^0$ matrix-valued function $P : \mathbf{X} \to \mathbb{R}^{n\times n}$ with $P(x) = P^T(x) \geq 0$, such that $\mathcal{M}(P(x),x) \leq 0$ for all $x \in \mathbf{X}$.*

Proof Since the set-valued function $\mathcal{F} : \mathbf{X} \rightsquigarrow \mathbf{P}(\mathbb{R}^{n\times n})$ is defined as

$$\mathcal{F}(x) = \{P : \mathcal{M}(P,x) \leq 0\},$$

by lemmas 6.1 and 6.2, $\mathcal{F}$ takes closed, convex sets as its values and is lower semi-continuous. Therefore, Michael's selection theorem (see lemma 8.7) applies, and there is a continuous selection $P : \mathbf{X} \to \mathbf{P}(\mathbb{R}^{n\times n})$ from $\mathcal{F}$, i.e. $\mathcal{M}(P(x),x) \leq 0$ for all $x \in \mathbf{X}$. □

6.2 Construction of Continuous Solutions to NLMIs

The computational issue of NLMIs can be pursued in terms of the techniques for solving LMIs, which can be solved in terms of convex optimization methods [3]. In this section, we demonstrate how this works. Let $\mathbf{X}$ be an open subset $\mathbb{R}^n$ with $0 \in \mathbf{X}$. In this subsection, we consider the NLMIs discussed in the preceding subsection (43):

$$\mathcal{M}(P,x) \leq 0.$$

with $x \in \mathbf{X}$, where $\mathcal{M} : \mathbf{P}(\mathbb{R}^{n\times n}) \times \mathbf{X} \to \mathbf{S}(\mathbb{R}^{m\times m})$ is continuous and satisfies (42), i.e.:

$$\mathcal{M}(\sum_{k=1}^{N} \alpha_k P_k, x) = \sum_{k=1}^{N} \alpha_k \mathcal{M}(P_k, x)$$

for all $\alpha_k \geq 0$ with $\sum_{k=1}^{N} \alpha_k = 1$.

6.2.1 Existence of Local Constant Solutions to NLMIs

In this subsection, we are considering the existence of local constant solutions to NLMIs. To be more concrete, we take the NLMI (9) as an example.

$$\begin{bmatrix} A^T(x)P(x) + P(x)A(x) & P(x)B(x) & C^T(x) \\ B^T(x)P(x) & -I & D^T(x) \\ C(x) & D(x) & -I \end{bmatrix} \leq (<)0.$$

where the coefficient matrix-valued functions $A(x), B(x), C(x), D(x)$ are assumed to be continuous on bounded open subset $\mathbf{N} \subset \mathbb{R}^n$. Denote the left hand side of the above NLMI as $\mathcal{M}(P, \mathcal{C}(x))$, where $\mathcal{C}$ is the coefficient function $\mathcal{C} : x \mapsto [A(x), B(x), C(x), D(x)]$ which is continuous on $\mathbf{N}$, so there are some constant matrices A_i, B_i, C_i, D_i with $I - D_i^T D_i \leq 0$ for $i \in \{1, 2, \cdots, L\}$ for some positive integer L, such that

$$\mathcal{C}(x) \in Co\{[A_i, B_i, C_i, D_i]|_{i\in\{1,2,\cdots,L\}}\}, \forall x \in \mathbf{N},$$

where Co stands for the **convex hull.** If there is a constant (semi-)positive definite matrix $P \in \mathbb{R}^{n\times}$ such that

$$\mathcal{M}(P, [A_i, B_i, C_i, D_i]) \leq (<)0, \forall i \in \{1, 2, \cdots, L\}, \tag{47}$$

i.e.,

$$\left[\begin{array}{ccc} A_i^T P + PA_i & PB_i & C_i^T \\ B_i^T P & -I & D_i^T \\ C_i & D_i & -I \end{array}\right] \leq (<) 0, \forall i \in \{1, 2, \cdots, L\},$$

which are a set of linear matrix inequalities (LMIs) and can be solved in terms of convex optimization methods [3], then P also satisfies

$$\mathcal{M}(P, \mathcal{C}(x)) \leq (<) 0.$$

for all $x \in \mathbf{N}$; i.e.,

$$\left[\begin{array}{ccc} A^T(x)P + PA(x) & PB(x) & C^T(x) \\ B^T(x)P & -I & D^T(x) \\ C(x) & D(x) & -I \end{array}\right] \leq (<) 0.$$

for all $x \in \mathbf{N}$. For detailed discussion about the solutions of LMIs, consult [3].

6.2.2 Construction of Continuous Solutions of NLMIs

Here, we again are not going to specify the forms of NLMIs. Let $\mathbf{X}$ be an open subset $\mathbb{R}^n$ with $0 \in \mathbf{X}$. Consider the NLMI (43):

$$\mathcal{M}(P, x) \leq 0.$$

with $x \in \mathbf{X}$, where $\mathcal{M} : \mathbf{P}(\mathbb{R}^{n \times n}) \times \mathbf{X} \rightarrow \mathbf{S}(\mathbb{R}^{m \times m})$ is continuous and satisfies (42), i.e.:

$$\mathcal{M}(\sum_{k=1}^{N} \alpha_k P_k, x) = \sum_{k=1}^{N} \alpha_k \mathcal{M}(P_k, x) \tag{48}$$

for all $\alpha_k \geq 0$ with $\sum_{k=1}^{N} \alpha_k = 1$.

The set valued functions $\mathcal{F}, \check{\mathcal{F}} : \mathbf{X} \rightarrow \mathbf{P}(\mathbb{R}^{n \times n})$ are defined as in (44) and (45). It is assumed that both $\mathcal{F}$ and $\check{\mathcal{F}}$ are strict, i.e., $\text{Dom}(\mathcal{F}) = \mathbf{X}$ and $\text{Dom}(\check{\mathcal{F}}) = \mathbf{X}$.

Since $\text{Dom}(\check{\mathcal{F}}) = M$, for each $x \in \mathbf{X}$, there is a semi-positive definite $P_x \in \mathbb{R}^{n \times n}$ such that

$$\mathcal{M}(P_x, x) < 0.$$

(Note that the continuity of M with respect to the first argument enables us to choose P_x to be positive definite). By continuity of M with respect to x, there is a $r_x > 0$ such that for all $x_0 \in \mathbf{N}(x) := \{x_0 : \|x_0 - x\| < r_x\}$,

$$\mathcal{M}(P_x, x_0) < 0. \tag{49}$$

(For practical computation, we can use the process introduced in section 6.2.1 to get the local solution for a given neighborhood $\mathbf{N}(x)$.)

On the other hand, $\{\mathbf{N}(x)\}|_{x \in \mathbf{X}}$ is an open covering of X, i.e.,

$$X \subset \bigcup_{x \in \mathbf{X}} \mathbf{N}(x) \tag{50}$$

Since the space $\mathbb{R}^n$ is *paracompact*, there is a locally finite open subcovering $\{\mathbf{N}_i\}|_{i\in\mathbf{I}}$ for some index set $\mathbf{I}$ which refines $\{\mathbf{N}(x)\}|_{x\in\mathbf{X}}$ [1, p.10]. By (49), $P_i \in \mathbb{R}^{n\times n}$ is taken to be (semi-)positive definite for each $i \in \mathbf{I}$ such that

$$\mathcal{M}(P_i, x) < 0. \tag{51}$$

for all $x \in \mathbf{N}_i$.

It is known by the standard results of **continuous partitions of unity** (see, for instance, [1, pp.9-11]) that there is a locally Lipschitzean partition of unity $\{\psi_i\}|_{i\in\mathbf{I}}$ to $\mathbf{X}$ subordinated to the covering $\{\mathbf{N}_i\}|_{i\in\mathbf{I}}$; i.e., ψ_i is locally Lipschitzean and non-negative with support $Supp(\psi_i) \subset \mathbf{N}_i$ for each $i \in \mathbf{I}$, and

$$\sum_{i\in\mathbf{I}} \psi_i(x) = 1, \forall x \in \mathbf{X}. \tag{52}$$

Define a matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{P}(\mathbb{R}^{n\times n})$ as

$$P(x) = \sum_{i\in\mathbf{I}} \psi_i(x) P_i, \forall x \in \mathbf{X}, \tag{53}$$

which is (semi-)positive definite and continuous since it is locally a finite sum of continuous (semi-)positive definite matrix-valued functions.

So it follows from (52), (53) and (48) that

$$\mathcal{M}(P(x), x) = \mathcal{M}(\sum_{i\in\mathbf{I}} \psi_i(x) P_i, x) = \sum_{i\in\mathbf{I}} \psi_i(x) \mathcal{M}(P_i, x) < 0$$

The last equality holds since the sum is finite for each $x \in \mathbf{X}$.

Thence, the constructed $\mathbf{C}^0$ matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{P}(\mathbb{R}^{n\times n})$ in (53) is (semi-)positive definite and is a solution to $\mathcal{M}(P(x), x) < 0$.

6.3 Remarks on Solvability of $\mathcal{H}_\infty$-Control Problems

As mentioned earlier, the existence of semi-positive definite matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{P}(\mathbb{R}^{n\times n})$ to NLMIs is not enough to gaurantee the strong $\mathcal{H}_\infty$-control problem to have solution; some additional requirement is imposed in this paper, i.e. there is a $\mathbf{C}^1$ function, which is called **storage function**, $V : \mathbf{X} \rightarrow \mathbb{R}^+$, such that

$$\frac{\partial V}{\partial x}(x) = 2x^T P(x)$$

for all $x \in \mathbf{X}$. In this subsection, we will examine explicitly when it is the case for the solution constructed in the preceding subsection.

Consider the NLMI (43):

$$\mathcal{M}(P, x) \leq 0$$

with $x \in \mathbf{X}$, where $\mathcal{M} : \mathbf{P}(\mathbb{R}^{n\times n}) \times \mathbf{X} \rightarrow \mathbf{S}(\mathbb{R}^{m\times m})$ is continuous and satisfies (42), i.e.:

$$\mathcal{M}(\sum_{k=1}^{N} \alpha_k P_k, x) = \sum_{k=1}^{N} \alpha_k \mathcal{M}(P_k, x)$$

for all $\alpha_k \geq 0$ with $\sum_{k=1}^{N} \alpha_k = 1$.

From the preceding subsection, a matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{P}(\mathbb{R}^{n\times n})$, which satisfies $\mathcal{M}(P, x) \leq 0$, is constructed as (53)

$$P(x) = \sum_{i\in\mathbf{I}} \psi_i(x) P_i, \forall x \in \mathbf{X},$$

for some index set $\mathbf{I}$, where $\{\psi_i\}|_{i\in\mathbf{I}}$ is a partition of unity of $\mathbf{X}$ and $P_i = P_i^T \geq 0$. We further assume $\mathbf{X} \subset \mathbb{R}^\mathbf{n}$ is bounded, then $\mathbf{I}$ can be chosen to be finite, i.e. $\mathbf{I} = \{1, 2, \cdots K\}$. As for each $i \in \mathbf{I}$, $\psi_i : \mathbf{X} \rightarrow \mathbb{R}^+$ is locally Lipschitzean, ψ_i is absolutely continuous on $\mathbf{X}$; $\frac{\partial \psi_i}{\partial x}(x)$ exists almost everywhere on $\mathbf{X}$. For clarity, we assume $\psi_i \in \mathbf{C}^1$. Now let $d\psi_i$ be the differential of ψ_i, it is a 1-form on $\mathbf{X}$, so

$$d\psi_i = \sum_{j=1}^{n} \frac{\partial \psi_i}{\partial x_j} dx_j, \forall i \in \mathbf{I} = \{1, 2, \cdots, K\}. \tag{54}$$

On the other hand, for each $i \in \mathbf{I}$, define a function $V_i : \mathbf{X} \rightarrow \mathbb{R}^+$ as

$$V_i(x) = x^T P_i x. \tag{55}$$

The differential dV_i of V_i also defines a 1-form on $\mathbf{X}$, and

$$dV_i = \sum_{l=1}^{n} \frac{\partial V_i}{\partial x_l} dx_l = 2x^T P_i dx, \forall i \in \mathbf{I} = \{1, 2, \cdots, K\}. \tag{56}$$

where $dx := \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$.

With the above preparation, we state the main result of this subsection as follows.

Theorem 6.4 *Suppose the matrix valued function $P : \mathbf{X} \rightarrow \mathbf{P}(\mathbb{R}^{n\times n})$ defined by*

$$P(x) = \sum_{i=1}^{K} \psi_i(x) P_i \tag{57}$$

with $\psi_i : \mathbf{X} \rightarrow \mathbb{R}^+$ being of class $\mathbf{C}^1$ and $P_i \in \mathbf{P}(\mathbb{R}^{n\times n})$ for $i \in \{1, 2, \cdots, K\}$ satisfies (43): $\mathcal{M}(P, x) \leq 0$ for all $x \in \mathbf{X}$; let $V_i(x) = x^T P_i x$ for all $i \in \{1, 2, \cdots, K\}$. There exists a $\mathbf{C}^2$ function $V : \mathbf{X} \rightarrow \mathbb{R}$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ if and only if

$$\sum_{i=1}^{K} \frac{\partial \psi_i}{\partial x_j}(x) \cdot \frac{\partial V_i}{\partial x_l}(x) = \sum_{i=1}^{K} \frac{\partial \psi_i}{\partial x_l}(x) \cdot \frac{\partial V_i}{\partial x_j}(x) \tag{58}$$

for all $x \in \mathbf{X}$ and $j, l \in \{1, 2, \cdots, n\}$ with $j \neq l$.

Proof Define a 1-form ω on $\mathbf{X}$ as follows

$$\omega := \sum_{i=1}^{K} \psi_i dV_i \tag{59}$$

So by (56), it follows that

$$\omega(x) = \sum_{i=1}^{K} 2\psi_i(x) x^T P_i dx = 2x^T P(x) dx$$

Thus, there is a $\mathbf{C}^2$ function $V : \mathbf{X} \to \mathbb{R}$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ if and only if $\omega = dV$, i.e. the 1-form defined by (59) is **exact**. Since the space $\mathbb{R}^n$ is contractible, by Poincare lemma (lemma 8.1), the later statement is equivalent to that the 1-form ω is **closed**. i.e. the 2-form $d\omega$ on $\mathbf{X}$, which is the differential of ω, is 0.

On the other hand,

$$d\omega = d\left(\sum_{i=1}^{K} \psi_i dV_i\right) = \sum_{i=1}^{K} d\psi_i \wedge dV_i$$

$$= \sum_{i=1}^{K} \sum_{j<l} \left(\frac{\partial \psi_i}{\partial x_j} \cdot \frac{\partial V_i}{\partial x_l} - \frac{\partial \psi_i}{\partial x_l} \cdot \frac{\partial V_i}{\partial x_j}\right) dx_j \wedge dx_l \tag{60}$$

$$= \sum_{j<l} \sum_{i=1}^{K} \left(\frac{\partial \psi_i}{\partial x_j} \cdot \frac{\partial V_i}{\partial x_l} - \frac{\partial \psi_i}{\partial x_l} \cdot \frac{\partial V_i}{\partial x_j}\right) dx_j \wedge dx_l. \tag{61}$$

Where (60) is derived by substituting $d\psi_i$ and dV_i from (54) and (56) and then re-organizing it by using that $dx_l \wedge dx_j = -dx_j \wedge dx_l$ (so $dx_j \wedge dx_j = 0$) for $j, l \in \{1, 2, \cdots, n\}$.

From (61) and the linear independence of 2-forms $\{dx_j \wedge dx_l\}|_{j<l}$, it follows that $d\omega = 0$ if and only if

$$\sum_{i=1}^{K} \left(\frac{\partial \psi_i}{\partial x_j} \cdot \frac{\partial V_i}{\partial x_l} - \frac{\partial \psi_i}{\partial x_l} \cdot \frac{\partial V_i}{\partial x_j}\right) = 0, \forall j, l \in \{1, 2, \cdots, n\}, j < l,$$

which is equivalent to (58). □

7 Concluding Remarks

In this paper, we have characterized the $\mathcal{H}_\infty$-control problem for a class of nonlinear systems in terms of nonlinear matrix inequalities which result in the convex problems. Unfortunately, unlike the linear case, the solution of the NLMIs by themselves are not sufficient to guarantee the existence of the required controller. However, the proposed approach indeed points out a new direction to make the nonlinear $\mathcal{H}_\infty$-control theory to be applicable.

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8 Appendix: Some Technical Results

8.1 Equation $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$

In this subsection, we shall strictly treat and clarify some issue related the equation:

$$\frac{\partial V}{\partial x}(x) = 2x^T P(x) \tag{62}$$

where $P = P^T : \mathbf{X} \to \mathbb{R}^{n \times n}$ and $V : \mathbf{X} \to \mathbb{R}$. $\mathbf{X}$ is an open convex subset in $\mathbb{R}^n$ containing the origin. Unless otherwise stated, it is assumed that $P \in \mathbf{C}^0$ and $V \in \mathbf{C}^1$. In addition, $V \in \mathbf{C}^1$ satisfies (62) if and only if $V(x) = x^T Q x + r(x)$ with some $Q = Q^T \in \mathbb{R}^{n \times n}$ and $\mathbf{C}^1$ function $r : \mathbf{X} \to \mathbb{R}$ satisfying

$$\lim_{x \to 0} \frac{|r(x)|}{\|x\|^2} = 0.$$

We first characterize a class of matrix-valued function which satisfies (62) with some function $V : \mathbf{X} \to \mathbb{R}$. The derivation depends on the following lemma which is known as Poincare lemma (see [25, p.306]).

Lemma 8.1 *If $\mathbf{M}$ is a smoothly contractible manifold, then every closed form on $\mathbf{M}$ is exact.*

Lemma 8.2 *Suppose a matrix-valued function $P : \mathbf{X} \to \mathbb{R}^{n \times n}$ is of class $\mathbf{C}^1$; let*

$$P(x) = [p_1(x)\ p_2(x)\ \cdots\ p_n(x)],$$

define $v_i(x) = x^T p_i(x)$ for $i = 1, 2, \cdots, n$ and $x \in \mathbf{X}$. Then there exists $V : \mathbf{X} \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(x) = 2x^T P(x)$$

if and only if

$$\frac{\partial v_i}{\partial x_j}(x) = \frac{\partial v_j}{\partial x_i}(x). \tag{63}$$

for all $i, j = 1, 2, \cdots, n$.

Proof Since $v_i : x \mapsto x^T p_i(x)$ defines a differentiable function on $\mathbf{X}$, so its differential, dv_i, defines a 1-form on $\mathbf{X}$, and

$$dv_i = \sum_{j=1}^{n} \frac{\partial v_i}{\partial x_j} dx_j$$

On the other hand, define a 1-form as

$$\omega = \sum_{i=1}^{n} v_i dx_i = 2x^T p(x) dx$$

where $dx := \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$.

Thence, there exists $V : \mathbf{X} \rightarrow \mathbb{R}$ such that $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ if and only if $\omega = dV$, i.e. the 1-form defined above is **exact**. Since the space $\mathbb{R}^n$ is **contractible**, by **Poincare Lemma**, the later statement is equivalent to that the 1-form ω is **closed**. i.e. the 2-form $d\omega$ on $\mathbf{X}$, which is the differential of ω, is 0.

On the other hand,

$$d\omega = d(\sum_{i=1}^{n} v_i dx_i) = \sum_{i=1}^{n} dv_i \wedge dx_i$$

$$= \sum_{i=1}^{n} (\sum_{j=1}^{n} \frac{\partial v_i}{\partial x_j} dx_j) \wedge dx_i$$

$$= \sum_{j<i} (\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}) dx_j \wedge dx_i$$

Where the last equality is derived by using that $dx_i \wedge dx_j = -dx_j \wedge dx_i$ (so $dx_j \wedge dx_j = 0$) for $i, j \in \{1, 2, \cdots, n\}$.

From the linear independence of 2-forms $\{dx_j \wedge dx_l\}|_{j<l}$, it follows that $d\omega = 0$ if and only if

$$\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} = 0, \forall i, j \in \{1, 2, \cdots, n\}, j < i,$$

which is equivalent to (63). □

Lemma 8.3 *Let $V : \mathbf{X} \rightarrow \mathbb{R}$ with $V(0) = 0$ be such that*

$$\frac{\partial V}{\partial x}(x) = 2x^T P(x)$$

for some $P = P^T : \mathbf{X} \rightarrow \mathbb{R}^{n \times n}$. If $P(x) > 0$ for all $x \in \mathbf{X}$, then $V(x) > 0$ for all $x \in \mathbf{X} \setminus 0$; if $P(x) \geq 0$ for all $x \in \mathbf{X}$, then $V(x) \geq 0$ for all $x \in \mathbf{X}$.

Proof We prove the former statement here, the latter statement is proved similarly. Suppose $V(x_0) \leq 0$ for some $x_0 \in \mathbf{X}$ and $x_0 \neq 0$.

Since $V \in \mathbf{C}^1$, by the mean-value theory, there exists $k \in (0, 1)$ such that

$$V(x_0) - V(0) = \frac{\partial V}{\partial x}(kx_0)(x_0 - 0) = 2kx_0^T P(kx_0)x_0,$$

as $V(x_0) - V(0) \leq 0$, the above equality implies

$$x_0^T P(kx_0) x_0 \leq 0. \tag{64}$$

On the other hand, since $P(kx_0) > 0$ by assumption, $x_0 \neq 0$, so

$$x_0^T P(kx_0) x_0 > 0. \tag{65}$$

Note that (64) and (65) lead to a contradiction. This completes the proof. □

8.2 Schur Complements

A reference for the material here is [11].

Lemma 8.4 *Suppose $M = M^T \in \mathbb{R}^{(n+m)\times(n+m)}$ is partioned as*

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

with $C \in \mathbb{R}^{m+m}$ is non-singular, then $P \geq 0$ if and only if $C > 0$ and $A - BC^{-1}B^T \geq 0$.

Lemma 8.5 *Let $X = X^T, Y = Y^T \in \mathbb{R}^{n\times n}$ be two positive definite matrices. Then there is a positive definite matrix $P = P^T \in \mathbb{R}^{(n+m)\times(n+m)}$ such that*

$$P = \begin{bmatrix} X & X_1^T \\ X_1 & X_0 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} Y & Y_1^T \\ Y_1 & Y_0 \end{bmatrix}$$

if and only if $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0.$

8.3 Characterization of a State-Dependent LMI

Consider a matrix-valued function $B : \mathbf{M} \to \mathbb{R}^{m\times n}$, with $m \leq n$.

$$B(x) = \begin{bmatrix} b_1(x) \\ \vdots \\ b_m(x) \end{bmatrix}$$

with $b_1(x), b_2(x), \cdots, b_m(x)$ are (smooth) co-vector fields. They span a (smooth) **co-distribution**:

$$\Omega(B(x)) := span\{b_1(x), b_2(x), \cdots, b_m(x)\} \tag{66}$$

It is assumed that each $x \in \mathbf{M}$ is a regular point of $\Omega(B(x))$, and the dimension of the co-distribution $\dim(\Omega(B(x))) = m$; thus, there is an $(n-m)$-dimensional (smooth) **distribution** $\mathcal{N}(B(x))$ which is the annihilator of $\Omega(B(x))$, i.e.

$$\mathcal{N}(B(x)) := \{v \in \mathbb{R}^n :< w^*, v >= 0, \forall w^* \in \Omega(B(x))\} \tag{67}$$

Moreover, there is a (smooth) matrix-valued function $B_\perp : \mathbf{M} \to \mathbb{R}^{n\times(n-m)}$, such that its columns, which are (smooth) vector fields on $\mathbf{M}$, span the distribution $\mathcal{N}(B(x))$, i.e. $\mathcal{N}(B(x)) = span(B_\perp(x))$ for $x \in \mathbf{M}$. The reader is referred to [12] for more detailed introduction about distributions and co-distributions.

The following lemma follows from [4] (see for example, [24, 9, 15]).

Lemma 8.6 *Consider the following matrix inequality*

$$Q(x) + U^T(x)F^T(x)V(x) + V^T(x)F(x)U(x) \leq 0 \tag{68}$$

with $Q = Q^T : \mathbf{M} \to \mathbb{R}^{m \times m}$, $U : \mathbf{M} \to \mathbb{R}^{r \times m}$ *with* $dim(\Omega(U(x)) = r < m$, *and* $V : \mathbf{M} \to \mathbb{R}^{s \times m}$ *with* $dim(\Omega(V(x)) = s < m$, *then (68) has a solution* $F : \mathbf{M} \to \mathbb{R}^{s \times r}$ *if and only if*

$$U_\perp^T(x)Q(x)U_\perp(x) \leq 0, \qquad V_\perp^T(x)Q(x)V_\perp(x) \leq 0$$

for some $U_\perp : \mathbf{M} \to \mathbb{R}^{m \times (m-r)}$ *such that* $span(U_\perp(x) = \mathcal{N}(U(x))$ *and* $V_\perp : \mathbf{M} \to \mathbb{R}^{m \times (m-s)}$ *such that* $span(V_\perp(x) = \mathcal{N}(V(x))$.

8.4 Set-Valued Maps and Their Selections

A reference for the material here is [1].

Let $\mathcal{X}$ and $\mathcal{Y}$ be two sets. A **set valued map** F from $\mathcal{X}$ to $\mathcal{Y}$ is a map that associates with any $x \in \mathcal{X}$ a subsect $F(x)$ of $\mathcal{Y}$. We denote it as

$$F : \mathcal{X} \rightsquigarrow \mathcal{Y}.$$

The subsets $F(x)$ are called the **values** of F. The **domain** of F is defined as

$$\mathrm{Dom}(F) := \{x \in \mathcal{X} : F(x) \neq \emptyset\}.$$

The map is said to be **strict** if $\mathrm{Dom}(F) = \mathcal{X}$.

Definition 8.1 *The set-valued map* $F : \mathcal{X} \rightsquigarrow \mathcal{Y}$ *is said to be* **lower semi-continuous** *at* $x_0 \in \mathcal{X}$ *if for any* $y \in F(x_0)$ *and any neighborhood* $\mathrm{N}(y)$ *of* y, *there exists a neighborhood* $\mathrm{N}(x_0)$ *of* x_0 *such that for all* $x \in \mathrm{N}(x_0)$, $F(x) \cap \mathrm{N}(y) \neq \emptyset$. F *is said to be lower semi-continuous if it is lower semi-continuous at every* $x_0 \in \mathcal{X}$.

When $\mathcal{X}$ and $\mathcal{Y}$ are metric spaces, the above definition can be phrased as follows: *Given any sequence* $\{x_n\} \subset \mathcal{X}$ *converging to* $x_0 \in \mathcal{X}$ *and any* $y_0 \in F(x_0)$, *there exists a sequence* $\{y_n\}$ *with* $y_n \in F(x_n)$ *that converges to* y_0. Unlike the single-valued maps, the above definition is not equivalent to the following statement: *For any open subset* N *of* $\mathcal{Y}$ *containing* $F(x_0)$, *there exists a neighborhood* $\mathrm{N}(x_0)$ *of* x_0 *such that* $F(\mathrm{N}(x_0)) \subset \mathrm{N}$. Actually the latter statement defines **upper semi-continuity** of set-valued maps.

Given a set-valued map $F : \mathcal{X} \rightsquigarrow \mathcal{Y}$, it is known from the **Axiom of Choice** that, there is map $f : \mathcal{X} \to \mathcal{Y}$ which is a **selection** of F, i.e. $f(x) \in F(x)$ for each $x \in \mathcal{X}$. For a class of set-valued maps, we have the following lemma which is known as **Michael's selection theorem** (cf [1]).

Lemma 8.7 *Let* $\mathcal{X}$ *be a metric space,* $\mathcal{Y}$ *a Banach space,* $F : \mathcal{X} \rightsquigarrow \mathcal{Y}$ *which has the closed convex subsets as its values be lower semi-continuous. Then there exists a continuous selection* $f : \mathcal{X} \to \mathcal{Y}$ *from* F.

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